Higher Gauge Theory, Self-Dual Strings and 6D Superconformal Field Theory

Lennart Schmidt

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Heriot-Watt University

Department of Mathematics,
School of Mathematical and Computer Sciences.

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Abstract

We present two explicit constructions in higher gauge theory of relevance to string and M-theory: the non-abelian self-dual string and a six-dimensional (1,0) superconformal field theory.

We start by outlining higher gauge theory from the point of view of morphisms of graded differential algebras and extend this to generalized higher gauge theory. We discuss two models of the string Lie 2-algebra and give twisted versions of these that are suitable for our non-abelian constructions.

We argue from analogy to monopoles that the string Lie 2-algebra is the relevant higher gauge structure for the non-abelian generalization of the self-dual string. We show that the twisted versions can be used to write down consistent non-abelian self-dual string equations. Moreover, we give the elementary solution, which passes the relevant consistency checks.

We also use this gauge structure to present an action for a six-dimensional superconformal field theory containing a non-abelian tensor multiplet based on ingredients available in the literature. The resulting (1,0)-model contains the field content of the (2,0)-theory, allows for a self-dual three-form curvature and straightforwardly reduces to a four-dimensional supersymmetric Yang–Mills theory. It can be regarded as a stepping stone towards a potential construction of the (2,0)-theory.

To my parents and my sister.

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Chapter 1

Introduction

1.1 On parallel transport and higher gauge theory

The main framework this thesis' work is based on is a natural generalization of gauge theory known as *higher gauge theory*. Dating back to the discovery of gauge invariance of classical electrodynamics in the 19^{th} century, ordinary gauge theory is now ubiquitous in modern physics. Most importantly, it serves as the basis of the quantum field theories underlying the standard model and is, as such, vital to our theoretical understanding of nature.

In geometrical terms, the potential of a gauge theory is described by a connection living on a principal G-bundle, where G is a Lie group, and, thus, it describes the parallel transport of pointlike particles along one-dimensional paths. Higher gauge theory is the analogue for extended objects and their parallel transport along higher dimensional surfaces [1–3]. Similarly to the importance of ordinary gauge theory for pointlike particles, higher gauge theory should, therefore, be of central interest in string and M-theory, as these deal with extended objects.

Let us discuss parallel transport in a little more detail. Pictorially we can write

$$\begin{tikzcd} \bullet \arrow[r, bend left, "g"] & \bullet \end{tikzcd} \tag{1.1}$$

symbolizing a zero-dimensional particle parallel transported along a one-dimensional path [2], which is naturally associated with a Lie group element $g \in \mathsf{G}$. Subsequent transportation along two paths corresponds to the group product and the requirement that the triple composite

$$\begin{tikzcd} \bullet \arrow[r, bend left, "g"] & \bullet \arrow[r, bend left, "g_1"] & \bullet \arrow[r, bend left, "g_2"] & \bullet \end{tikzcd} \tag{1.2}$$

is well defined tells us that the group product is associative. We can make this more precise by using category theory as a natural framework for describing the parallel transport of particles. The assignment of group elements g to paths as in (1.1) is made explicit in smooth functors

$$\mathrm{hol} : \mathcal{P}_1(M) \to \mathsf{G} \ , \tag{1.3}$$

where G is seen as a one-object category and $\mathcal{P}_1(M)$ is the path groupoid of a manifold M, which is the category of objects given by the points of M and morphisms given by the paths between them[1]. These functors are in one-to-one correspondence with connections on the trivial G-bundle over M [1].

Similarly, consider the parallel transport of one-dimensional strings along a two-dimensional surface

$$\begin{tikzcd} \bullet \arrow[r, bend left=50] \arrow[r, bend right=50] \arrow[r, phantom, "\Downarrow g"] & \bullet \end{tikzcd} \tag{1.4}$$

associated with a group element g. Now, the different ways of composing the parallel transports in the diagram

$$\begin{tikzcd} \bullet \arrow[r, bend left=50, ""{name=A, below}] \arrow[r, ""{name=B, above}, ""{name=C, below}] \arrow[r, bend right=50, ""{name=D, above}] \arrow[from=A, to=B, Rightarrow, "g_1"] \arrow[from=C, to=D, Rightarrow, "g_2"] & \bullet \arrow[r, bend left=50, ""{name=E, below}] \arrow[r, ""{name=F, above}, ""{name=G, below}] \arrow[r, bend right=50, ""{name=H, above}] \arrow[from=E, to=F, Rightarrow, "g_1'"] \arrow[from=G, to=H, Rightarrow, "g_2'"] & \bullet \end{tikzcd} \tag{1.5}$$

should agree. In equations, this means that

$$(g_2 g_1)(g_2' g_1') = (g_2 g_2')(g_1 g_1') \ , \tag{1.6}$$

which forces G to be abelian, by an argument due to Eckmann and Hilton [4]. This argument was also later rediscovered, in infinitesimal form, e.g. in [5]. Indeed,

[1]There are some technicalities regarding the smoothness at the endpoints which we will suppress here.

setting $g_2 = 1$ and $g_1' = 1$ in the above equation immediately yields $g_1 g_2' = g_2' g_1$. This argument remains valid even if we introduce different compositions as long as both of these have an identity.

One would, however, want to be able to consider non-abelian higher gauge theory as will become clear later. This naturally leads to the concept of 2-categories, where we have 1-morphisms, denoted by $\rightarrow$, together with 2-morphisms, denoted by $\Rightarrow$, which are morphisms between morphisms. There are two ways of composing such 2-morphisms: horizontally, denoted by $\otimes$, and vertically, denoted by $\circ$. This yields a well-known generalization of the above equation known as the interchange law

$$(g_2 \circ g_1) \otimes (g_2' \circ g_1') = (g_2 \otimes g_2') \circ (g_1 \otimes g_1') \ , \tag{1.7}$$

based on the diagram

$$\begin{tikzcd} \bullet \arrow[r, bend left=50, "f_1"] \arrow[r, "f_2"] \arrow[r, bend right=50, "f_3"'] \arrow[r, phantom, shift left=3, "\Downarrow g_1"] \arrow[r, phantom, shift right=3, "\Downarrow g_2"] & \bullet \arrow[r, bend left=50, "f_1'"] \arrow[r, "f_2'"] \arrow[r, bend right=50, "f_3'"'] \arrow[r, phantom, shift left=3, "\Downarrow g_1'"] \arrow[r, phantom, shift right=3, "\Downarrow g_2'"] & \bullet \end{tikzcd} \ . \tag{1.8}$$

Crucially, the 2-morphisms now have sources and targets on which the compositions $\otimes$ and $\circ$ depend. This relaxes the Eckmann–Hilton argument and we can satisfy (1.7) even in the non-abelian case. Analogously to before, parallel transport is now given by smooth 2-functors [1]

$$\mathrm{hol} : \mathcal{P}_2(M) \rightarrow \mathcal{G} \ , \tag{1.9}$$

where $\mathcal{G}$ is a Lie 2-group and $\mathcal{P}_2(M)$ is the path 2-groupoid, which is obtained from $\mathcal{P}_1(M)$ by adding as 2-morphisms the surfaces enclosed by paths.

Thus, the study of parallel transport of extended objects naturally leads to the use of category theory. Indeed, many concepts from category theory appear in higher gauge theory. Among those, the ones of most relevance to this thesis are categorified Lie algebras and categorified bundles. The process of categorification is not uniquely defined and applies to many concepts in mathematics, but in the cases at hand, one, broadly speaking, replaces spaces by categories and equations by natural isomorphisms. For a Lie algebra this leads to the notion of weak and, in particular, semi-strict Lie 2-algebras [6], while for principal G-bundles it leads to principal $\mathcal{G}$-2-bundles [7], which are one of several equivalent ways of considering higher bundles [8].

Higher connections on such principal $\mathcal{G}$-2-bundles involve a 1-form A and a 2-form B taking values in a semi-strict Lie 2-algebra, which are equivalently described by 2-term L_∞-algebras [6]. Such 2-term L_∞-algebras offer a straightforward generalization to n-term L_∞-algebras, allowing for the consistent description of generalized connections involving forms of up to degree n.

Such higher connection forms play an important role in string and M-theory. For instance, the Kalb–Ramond 2-form field B is recognized to be part of the connective structure of a higher bundle [9, 10]. Thus, the study of higher gauge theory is of great interest and worthy of further investigation. More specifically, there is an elegant and useful approach to describing higher gauge theory based on morphisms of n-term L_∞-algebras, which is based on ideas going back to Cartan [11, 12] and Atiyah [13] and is fully given in [14]. The aim of this thesis is to use this rich framework to construct explicit non-abelian examples of higher gauge theory of relevance to string and M-theory, as the lack of such examples has been a shortcoming of higher gauge theory in the past.

1.2 The (2,0)-theory in six dimensions

One area of interest where higher gauge theory finds application is that of superconformal field theories. A conformal field theory on $\mathbb{R}^{q,p}$ is invariant under an action of the conformal algebra $\mathfrak{so}(p+1, q+1)$ and a superconformal theory has to be invariant under an action of a super extension of this Lie algebra. The simple, finite dimensional and complex Lie superalgebras have been classified [15, 16] and this list can be used to identify the possible super extensions of $\mathfrak{so}(p+1, q+1)$. Such extensions only exists for dimensions $p + q \leq 6$ as shown in [17], see also [18, 19]. Explicit examples of conformal and superconformal field theories have been known for a long time, and it was suspected that four was the maximal dimension for non-trivial unitary conformal field theories.

However, string and M-theory strongly suggest that there should also be a non-trivial six-dimensional superconformal field theory. A first such suggestion appeared in [20] where type IIB superstring theory on $\mathbb{R}^{1,5}\times$K3 is considered. When further compactified along an S^1, T-duality tells us that close to isolated singularities of the K3, massive strings wrapping the circle need to arise. In $\mathbb{R}^{1,5} \times \mathrm{K3}$ these correspond to self-dual D3-branes wrapping degenerating 2-cycles, which are called self-dual strings. The singularities in K3 are classified by diagrams of types A, D and E and when approaching these, the self-dual strings become massless and decouple from gravity suggesting there is a self-contained and consistent field theory describing the self-dual string in six dimensions.

There is a corresponding M-theory interpretation of self-dual strings in terms of boundaries of M2-branes ending on parallel M5-branes, where the former mediate the interactions of the latter [21, 22]. The M5-branes approaching each other corresponds to approaching the singularities where the self-dual strings become massless. At these singularities, the theory is argued to be a superconformal field theory in six dimensions [23]. For $\mathbb{R}^{q,p} = \mathbb{R}^{1,5}$ the above super-extension of the conformal algebra $\mathfrak{so}(6,2)$ leads to the superconformal Lie algebra $\mathfrak{osp}(6,2|4)$, which governs the symmetries of the system and contains the R-symmetry Lie algebra $\mathfrak{so}(5) \cong \mathfrak{sp}(2)$. This is the superconformal Lie algebra of the $\mathcal{N} = (2,0)$-theory describing the worldvolume of a stack of N M5-branes, simply known as *the (2,0)-theory*. For $N = 1$ this corresponds to the abelian case and the associated theory is given an explicit form in [24]. For general N however, a more concrete description is not available.

From supersymmetry we know that the field content of the (2,0)-theory is given by the (2,0)-tensor multiplet [18], which contains a self-dual 3-form field strength, four chiral spinors and five scalars. The latter can be regarded as the Goldstone scalars for the breaking of the symmetry group $\mathsf{SO}(10,1)$ of the full theory on $\mathbb{R}^{1,10}$ to $\mathsf{SO}(5,1) \times \mathsf{SO}(5)$ due to the presence of a flat M5-brane. Correspondingly, they describe the position of the M5-brane in the five directions orthogonal to the M5-brane's worldvolume. The degrees of freedom should scale as $\mathcal{O}(N^3)$ with the number N of M5-branes.

The (2,0)-theory plays an important role in M-theory since many dualities in string theory have their origin in different compactifications of the (2,0)-theory. For instance, reducing on a manifold $M_6 = M_4 \times E$, where E is an elliptic curve, yields $\mathcal{N} = 4$ gauge theory with modular parameter τ determined by E, which makes manifest the S-duality of the four-dimensional Yang–Mills theory [20, 25].

Furthermore, there is considerable scope for applications of the (2,0)-theory within mathematics, see e.g. [26]. An explicit description would, therefore, deepen our understanding and there are many approaches to furthering our knowledge of the (2,0)-theory, see e.g. [27–36].

The observables of the (2,0)-theory include Wilson surfaces [37], see also [38, 39]. This suggests that a classical description underlying the (2,0)-theory should be a theory of parallel transport of self-dual strings and as such a higher gauge theory. Furthermore, M2-brane models have been successfully constructed in [40–42] and been shown to be higher gauge theories [43, 44]. This gives hope that one can learn more about the (2,0)-theory from higher gauge theory.

There are a number of objections to the existence of a Lagrangian description of the (2,0)-theory. First, as the (2,0)-theory is a conformal theory it does not have any dimensionful parameters. Moreover, the singularities in the moduli space at which the self-dual strings become massless are isolated. Therefore, there are no continuous dimensionless parameters either. In the same vein, it can be shown that there are no continuous deformations of the abelian theory and, thus, no continuous parameter that can act as a coupling constant [45, 46]. This suggests there is no classical limit and therefore no Lagrangian description. However, note that very similar arguments are valid for the M2-brane models. There, a discrete parameter $k \in \mathbb{N}$ arose from an orbifold $\mathbb{C}^4/\mathbb{Z}_k$, circumventing the lack of continuous dimensionless parameters. One can hope an analogous argument to apply to the case of M5-branes.

Secondly, the existence of a Lagrangian description is made implausible by an argument from dimensional reduction [25]. We know that the six-dimensional (2,0)-theory should reduce to maximally supersymmetric Yang–Mills theory in five dimensions after compactification on a circle of radius R, which leads to a volume form $2\pi R\,\mathrm{d}^5x$ in the action. On the other hand, conformal invariance in six dimensions as well as dimensional analysis of the Yang–Mills term in the Lagrangian requires a volume form $\frac{1}{R}\,\mathrm{d}^5x$. This is a valid point. However, we will see that a direct and classical reduction to four dimensions can still be performed. Together with some dimensional oxidation, this might lead to the five-dimensional Yang–Mills theory in an indirect fashion.

In any case, the question of how much one can learn about the (2,0)-theory from higher gauge theory and how far one can go in constructing a Lagrangian is still of interest. To investigate this, let us discuss some of the features one would expect of such a Lagrangian description.

First, note that we would like to have the full field content of the (2,0)-tensor multiplet, but we do not necessarily expect full (2,0) supersymmetry. This is suggested by striking the parallel to the M2-brane models: these are not generally $\mathcal{N} = 8$ supersymmetric models but rather only exhibit $\mathcal{N} = 6$ supersymmetry. Thus we should merely expect $\mathcal{N} = (1,0)$ supersymmetric M5-brane models and not necessarily ones with full $\mathcal{N} = (2,0)$ supersymmetry — a point that was previously observed in [47].

Furthermore, one would like to be able to reduce the model to four-dimensional Yang–Mills theory as mentioned above. Additionally, there may be a reduction of the (2,0)-theory to the M2-brane models. In such a reduction, one would like to explain in particular the origin of the discrete Chern–Simons coupling.

As in the case of supersymmetric Yang–Mills theories, interesting classical configurations of the (2,0)-theory would be given by BPS states, which we should expect also to feature in our classical Lagrangian description.

Thus, together with the general discussion above we arrive at the following wishlist of items an interesting Lagrangian should, at minimum, satisfy:

(*1*) The action should contain an interacting 2-form gauge potential with self-dual curvature 3-form.

(*2*) The action should be based on solid mathematical foundations in order to allow for a formulation on general manifolds.

(*3*) The action should have the same field content and moduli space as the $(2,0)$-theory and be at least $\mathcal{N} = (1,0)$ supersymmetric.

(*4*) The gauge structure should arise from Lie algebras of types A, D and E.

(*5*) There should be a restriction of the action to that of a free $\mathcal{N} = (2,0)$ tensor multiplet.

(*6*) The action should have a self-dual string soliton as a BPS state, ideally the one of Chapter 4.

(*7*) There should be an appropriate reduction mechanism to four-dimensional super Yang–Mills theory, yielding gauge Lie algebras of types A, D and E.

(8) Ideally, there should be a reduction mechanism to three-dimensional M2-brane models explaining the origin of their discrete Chern–Simons coupling constant.

1.3 Outline and main results

We start in Chapter 2 by introducing the mathematical tools frequently used throughout the remainder of the thesis. We introduce L_∞-algebras $\mathfrak{g}$ from three different viewpoints: as a vector space with multilinear brackets μ_i, as a graded co-algebra with co-derivation $\mathcal{D}$ and as a graded algebra with differential Q. We focus on this last viewpoint, which is known as the Chevalley–Eilenberg algebra $\mathsf{CE}(\mathfrak{g})$, and recap the associated Weil algebra $\mathsf{W}(\mathfrak{g})$ and algebra of invariant polynomials $\mathsf{inv}(\mathfrak{g})$. We also define morphisms, 2-morphisms as in [14] and cyclic structures for L_∞-algebras. Additionally, we introduce the example of the string Lie 2-algebra, which appears in two categorically equivalent guises: the skeletal model $\mathfrak{string}_{\rm sk}(\mathfrak{g})$ and the loop model $\mathfrak{string}_{\hat{\Omega}}(\mathfrak{g})$.

In Chapter 3 we use these mathematical tools to recap the framework of higher gauge theory from morphisms of differential graded algebras. Furthermore, as done in [14] we define $\mathfrak{g}$-connection objects and use these to recap the twist of the skeletal model $\mathfrak{string}_{\rm sk}(\mathfrak{g})$ to $\widehat{\mathfrak{string}}_{\rm sk}(\mathfrak{g})$. We slightly extend this to also give a twist of the loop model arriving at $\widehat{\mathfrak{string}}_{\hat{\Omega}}(\mathfrak{g})$ as first done in [48]. Lastly, we give an overview of generalized higher gauge theory following the ideas outlined in [49].

In Chapter 4 we discuss the non-abelian self-dual string summarizing the results of [48]. We argue that the twisted string algebras $\widehat{\mathfrak{string}}_{\rm sk}(\mathfrak{g})$ and $\widehat{\mathfrak{string}}_{\hat{\Omega}}(\mathfrak{g})$ are good candidates for the gauge structure of a consistent non-abelian self-dual string and use the framework of higher gauge theory to find suitable self-dual string equations. In the case of the twisted model $\widehat{\mathfrak{string}}_{\rm sk}(\mathfrak{su}(2))$ the equations read as

$$\mathcal{H} = *\mathrm{d}\Phi \qquad \text{and} \qquad \mathcal{F} = *\mathcal{F} \;, \tag{1.10}$$

where Φ is an abelian Higgs field, $\mathcal{F}$ denotes the 2-form curvature and $\mathcal{H}$ is the

3-form curvature. The expressions for these curvatures are given by

$$\mathcal{H} = \mathrm{d}B - (A, \mathrm{d}A) - \tfrac{1}{3}(A, [A, A]) \qquad \text{and} \qquad \mathcal{F} = \mathrm{d}A + \tfrac{1}{2}[A, A] \;, \tag{1.11}$$

where $[-,-]$ and $(-,-)$ denote the commutator and Killing form of $\mathfrak{su}(2)$, respectively. These equations are gauge invariant and behave well under the categorical equivalence between $\widehat{\mathfrak{string}}_{\mathrm{sk}}(\mathfrak{g})$ and $\widehat{\mathfrak{string}}_{\hat\Omega}(\mathfrak{g})$. That is, gauge equivalence classes of solutions are mapped to gauge equivalence classes of solutions when exchanging the skeletal model with the loop model and vice versa. We give the charge one solution to (1.10) based on the elementary $\mathsf{SU}(2)$-instanton. The resulting fields are non-singular over $\mathbb{R}^4$ and approach the abelian self-dual string at infinity.

In Chapter 5 we recap the results of [50], where we gave a Lagrangian for a six-dimensional $\mathcal{N} = (1,0)$ superconformal theory that satisfies many of the items on the above wish-list. This action is based on a (1,0)-model derived from tensor hierarchies in supergravity [47, 51], which has a gauge structure that can be interpreted as a higher gauge theory as shown in [52]. The field content of this (1,0)-model consists of a (1,0)-tensor multiplet as well as a (1,0)-vector multiplet, so that, in order to arrive at the field content of the (2,0)-theory the action needs to be extended by terms for a (1,0)-hypermultiplet. Additionally, to allow for a self-dual three-form curvature a PST-type extension [53, 54] of the action is desirable. The general coupling to hypermultiplets has been worked out in [55] and a PST-type extension of the bosonic part of the action was given in [56]. We extend this to the full supersymmetric case and combine these pieces from the literature to arrive at a suitable action.

In the special case of higher gauge Lie algebra $\widehat{\mathfrak{string}}_{\mathrm{ext}}(\mathfrak{su}(2))$ together with 4×4 hypermultiplets, our model has the following Lagrangian:

$$\begin{aligned}\mathcal{L} = &-2\partial_\mu\phi_r\partial^\mu\phi_s - 8\bar\chi_r\partial\!\!\!/\chi_s - \tfrac{1}{6}\mathcal{H}^r_{\mu\nu\kappa}\mathcal{H}^{\mu\nu\kappa}_s + \tfrac{1}{6}\mathcal{H}^s_{\mu\nu\kappa}\,\mathrm{tr}\,(\bar\lambda\gamma^{\mu\nu\kappa}\lambda)\\ &-\phi_s\,\mathrm{tr}\left(F_{\mu\nu}F^{\mu\nu} - 2Y_{ij}Y^{ij} + 4\bar\lambda\nabla\!\!\!\!/\,\lambda\right) + 4\,\mathrm{tr}\,(\bar\lambda F_{\mu\nu})\gamma^{\mu\nu}\chi_s\\ &-16\,\mathrm{tr}\,(Y_{ij}\bar\lambda^i)\chi^j_s + \varepsilon^{\mu\nu\kappa\lambda\rho\sigma}\left(\tfrac{1}{36}C^r_{\mu\nu\kappa}\mathcal{H}^s_{\lambda\rho\sigma} + \tfrac{1}{8}B^s_{\mu\nu}\,\mathrm{tr}\,(F_{\kappa\lambda}F_{\rho\sigma})\right)\\ &-\;\mathrm{tr}\,(\nabla_\mu q\nabla^\mu q + 2\bar\psi\nabla\!\!\!\!/\,\psi - 8\bar\psi[\lambda, q] + 2q^i[Y_{ij}, q^j]) + \mathcal{L}_{\mathrm{PST}}\;.\end{aligned} \tag{1.12}$$

Here, we have two abelian tensor multiplets $(B_{r,s}, \chi^i_{r,s}, \phi_{r,s})$, an $\mathfrak{su}(2)$-valued vector multiplet (A, λ^i, Y^{ij}) and a non-dynamical abelian 3-form field C_r with curvature 2- and 3-forms

$$\mathcal{F} = \mathrm{d}A + \tfrac{1}{2}[A, A] \qquad \text{and} \qquad \mathcal{H} = \mathrm{d}B - (A, \mathrm{d}A) - \tfrac{1}{3}(A, [A, A]) + C_r \ . \tag{1.13}$$

We also have the hypermultiplets (q^i, ψ) taking values in the adjoint representation of $\mathfrak{su}(2)$. The explicit form of the PST term $\mathcal{L}_{PST}$ is found in (5.39).

This action has many of the properties in the above wish-list: It is a mathematically consistent formulation of an interacting 2-form potential and therefore it can be rather readily generalized to arbitrary spacetimes. It has the same field content as the full $(2,0)$-theory, but only $\mathcal{N} = (1,0)$ supersymmetry is realized. Moreover, the choice of gauge structure is rather natural and obtained from the considerations for the non-abelian self-dual string in Chapter 4. This action then also has this non-abelian self-dual string soliton as a classical BPS configuration. Additionally, if we set the $(1,0)$-vector multiplet to zero by choosing the higher gauge Lie algebra $\widehat{\mathfrak{string}}(*)$ and restrict the number of hypermultiplets, we recover the free abelian $\mathcal{N} = (2,0)$ action.

Furthermore, we show that our action for $\widehat{\mathfrak{string}}_{\mathrm{ext}}(\mathfrak{g})$ with $\mathfrak{g}$ a Lie algebra of type A, D or E allows for a straightforward reduction to $\mathcal{N} = 2$ super Yang–Mills theory in four dimensions with gauge Lie algebra $\mathfrak{g}$, where the modulus τ of the compactifying torus translates into the appropriate couplings, $\tau = \frac{\theta}{2\pi} + \mathrm{i}g_{\mathrm{YM}}^{-2}$. Also, a straightforward reduction to an M2-brane model in three dimensions is possible: our action for $\widehat{\mathfrak{string}}_{\mathrm{ext}}(\mathfrak{u}(n) \times \mathfrak{u}(n))$ turns into a supersymmetric deformation of the M2-brane model of [42].

However, our action does have a number of crucial, remaining issues. We comment on these, possible solutions and future directions in the conclusion.

Lastly, we include ecplicit formulas for morphisms and 2-morphisms in Appendix A and give more details of the calculation regarding our six-dimensional (1,0)-model in Appendix B.

Chapter 2

Mathematical Tools

In this chapter we introduce mathematical tools that will be relevant throughout the thesis. As such we will recall the notion of a Lie 2-algebra and the related notion of L_∞-algebras. Besides defining these, as usual, in terms of multi-brackets, we will also give two alternative points of view using differential graded co-algebras and differential graded algebras. This will lead to the definition of Chevalley–Eilenberg algebras together with the associated Weil algebra and invariant polynomials of an L_∞-algebra. Furthermore, we will also introduce the notion of 2-morphisms, first defined in [14]. The original references for L_∞-algebras are [57–59], further references include [6, 60–62] and useful reviews can be found in [1, 63]. We will also, thereafter, introduce the main example crucial to this thesis: the string Lie 2-algebra. Lastly, we discuss the notion of a cyclic structure for L_∞-algebras, mainly based on [64].

2.1 Categorified Lie algebras and L_∞-algebras

As seen in the introduction, categories appear naturally in the study of higher gauge theory. Let us, therefore, recall the basic definitions to fix notation.

Definition 2.1 (Category)

*A **category** $\mathscr{C}$ consists of*

- *a collection $\mathscr{C}_0$ of objects,*
- *a collection $\mathscr{C}_1$ of morphisms between objects, that is, for each $x, y \in \mathscr{C}_0$ a collection of morphisms $f : x \to y$,*
- *and, given two morphisms $f : x \to y$ and $g : y \to z$, a composition function*

$(g, f) \mapsto g \circ f,$

together with the following axioms:

- *the composition is associative, that is, given morphisms* $f : x \to y,\ g : y \to z$ *and* $h : z \to w$ *we have* $(h \circ g) \circ f = h \circ (g \circ f)$,
- *for any object* $x \in \mathscr{C}_0$ *there is an identity morphism* $\mathrm{id_x : x \to x} \in \mathscr{C}_1$ *such that* $\mathrm{id_y \circ f = f \circ id_x = f}$ *for any morphism* $f : x \to y$.

We often refer to source and target maps $s, t : \mathscr{C}_1 \to \mathscr{C}_0$ that map the morphisms to their respective start- and endpoints. That is, for a morphism $f : x \to y$ we write $s(f) = x$ and $t(f) = y$. Together with the identity assigning map $i : \mathscr{C}_0 \to \mathscr{C}_1$ given by $x \mapsto \mathrm{id_x}$, these maps define the structure of the category and are, as such, referred to as the structure maps — often symbolically written as $\mathscr{C} = \mathscr{C}_0 \leftleftarrows \mathscr{C}_1$.

We are furthermore interested in 2-morphisms between morphisms, which leads to the notion of a 2-category.

Definition 2.2 (2-category)

A ***2-category*** $\mathscr{C}$ *is a category further equipped with*

- *a collection of 2-morphisms* $\eta : f \Rightarrow g$ *for any two morphisms* $f, g : x \to y$,
- *a vertical composition* $\eta_1 \circ \eta_2$ *for 2-morphisms* $\eta_1 : f \Rightarrow g$ *and* $\eta_2 : g \Rightarrow h$,
- *a horizontal composition* $\eta \otimes \eta'$ *for 2-morphisms* $\eta : f \to g$ *and* $\eta' : f' \to g'$,

such that

- *both vertical and horizontal compositions are associative,*
- *given any morphism* $f : x \to y$ *there is an identity 2-morphism* $\mathrm{id_f : f \Rightarrow f}$ *with respect to vertical composition,*
- *there is an identity 2-morphism* $\mathrm{id_{id_x} : id_x \Rightarrow id_x}$ *for horizontal composition,*
- *and the interchange law*

$$(\eta_2 \circ \eta_1) \otimes (\eta_2' \circ \eta_1') = (\eta_2 \otimes \eta_2') \circ (\eta_1 \otimes \eta_1') \ , \tag{2.1}$$

is satisfied, cf. Equation (1.7).

There are more general notions for higher categories, see e.g. [65], however, the above definition will suffice and we will come back to the specific 2-morphisms relevant to this thesis in Section 2.4.

Let us then come to the categories central to this discussion: just as Lie algebras play a central role in ordinary gauge theory, categorified Lie algebras or, more generally, L_∞-algebras also play an essential role in higher gauge theory. As such they will feature prominently throughout this thesis. Following [6] we recall the definition of a semi-strict Lie 2-algebra and its relation to a 2-term L_∞-algebra. Let us first introduce a notion of a 2-vector space. For our purposes the straightforward definition of Baez–Crans [6] will suffice.

Definition 2.3 (2-vector space)

*A **2-vector space** is a category in which both the set of objects and the set of morphisms are vector spaces and the source, target, identity maps and composition of morphisms are all linear.*

Thus, 2-vector spaces are what are referred to as categories internal to the category of vector spaces.

Just as Lie algebras have an underlying vector space, Lie 2-algebras have an underlying 2-vector space so that we can use this to make the following definition:

Definition 2.4 (Semi-strict Lie 2-algebra)

*A **semi-strict Lie 2-algebra** is given by a 2-vector space L together with a bracket-functor $[\cdot,\cdot] : L \times L \to L$ which is*

- *bilinear, i.e. $[x, y]$ is linear in both arguments for all objects or morphisms $(x, y) \in L \times L$,*
- *skew-symmetric, i.e. $[x, y] = -[y, x]$ for all objects or morphisms $(x, y) \in L \times L$,*

and a natural isomorphism, the Jacobiator, $J_{x,y,z} : [[x, y], z] \to [[y, z], x] + [[z, x], y]$, which is

- *trilinear, i.e. $J_{x,y,z}$ is linear in the objects $x, y, z \in L$,*
- *completely anti-symmetric, i.e. its **arrow part** $\vec{J}_{x,y,z} = J_{x,y,z} - i(s(J_{x,y,z}))$ — where i and s are the identity-assigning and source maps — is anti-symmetric under permutations,*

and satisfies a higher coherence condition, called the Jacobiator identity, see [6] for details.

As such a semi-strict Lie 2-algebra is what one would expect from a categorification of a Lie algebra: spaces are replaced by categories and identities only need to hold up to natural isomorphisms. The term semi-strict refers to the fact that here the skew-symmetry of the bracket is still required to hold identically while the Jacobi identity is allowed to be violated in a controlled way. An important and closely related concept is that of an L_∞-algebra:

Definition 2.5 (L_∞-algebra)
An ***L_∞-algebra*** *or* ***strong homotopy Lie algebra*** $\mathfrak{g}$ *consists of*

- *a* $\mathbb{Z}$*-graded vector space* $\mathfrak{g} = \sum_{k\in\mathbb{Z}} \mathfrak{g}_k$*,*
- *and a collection of graded anti-symmetric, multilinear maps* $\mu_i : \wedge^i\mathfrak{g} \to \mathfrak{g}$ *of degree* $i-2$ *,*

which satisfy the ***higher*** *or* ***homotopy Jacobi relations***

$$\sum_{i+j=n}\sum_{\sigma\in S_{i|j}}(-1)^{ij}\chi(\sigma;x_1,\ldots,x_n)\mu_{j+1}(\mu_i(x_{\sigma(1)},\ldots,x_{\sigma(i)}),x_{\sigma(i+1)},\ldots,x_{\sigma(n)}) = 0 \tag{2.2}$$

for all $n\in\mathbb{N}$ *and* $x_1,\ldots,x_n\in\mathfrak{g}$*, where the second sum runs over all* ***(i,j)-unshuffles*** $\sigma\in S_{i|j}$*, which are the permutations satisfying* $\sigma(k)<\sigma(k+1)$ *for* $k\neq i$*, that is, permutations which map* $1,\ldots,n$ *to ordered lists of length* i *and* j*. Additionally,* $\chi(\sigma;x_1,\ldots,x_n)$ *denotes the* ***graded antisymmetric Koszul sign*** *defined by the graded antisymmetrized products*

$$x_1\wedge\cdots\wedge x_n = \chi(\sigma;x_1,\ldots,x_n)x_{\sigma(1)}\wedge\cdots\wedge x_{\sigma(n)}\ , \tag{2.3}$$

where any transposition involving at least one even degree elements acquires a minus sign. Finally, an **n*-term L_∞-algebra*** *is an* L_∞*-algebra that is concentrated, i.e. non-trivial only, in degrees* $0,\ldots,n-1$*.*

Explicitly, the graded anti-symmetry of the maps μ_i means that

$$\mu_i(x_{\sigma(1)},\ldots,x_{\sigma(n)}) = \chi(\sigma;x_1,\ldots,x_n)\mu_i(x_1,\ldots,x_n)\ , \tag{2.4}$$

where $x_1,\ldots,x_n\in\mathfrak{g}$ and σ is an arbitrary permutation.

Let us give a few illuminating examples: A 1-term L_∞-algebra is the same as an ordinary Lie algebra, as it is given by a single vector space $\mathfrak{g}_0$ of grading zero and the only non-trivial Jacobi relation is then given by

$$0 = \mu_2(\mu_2(x_1, x_2), x_3) - \mu_2(\mu_2(x_1, x_3), x_2) + \mu_2(\mu_2(x_2, x_3), x_1) \ , \tag{2.5}$$

where $x_1, x_2, x_3 \in \mathfrak{g}_0$. This is just the Jacobi identity with μ_2 being the commutator of the Lie algebra.

A 2-term L_∞-algebra, in turn, consists of two vector spaces $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ of degree 0 and 1, respectively. The lowest homotopy Jacobi relations are given by

$$\begin{aligned}
0 &= \mu_1\left(\mu_1\left(x_1\right)\right) \ , \\
0 &= (-1)^{|x_1||x_2|}\mu_2\left(\mu_1\left(x_2\right), x_1\right) + \mu_1\left(\mu_2\left(x_1, x_2\right)\right) - \mu_2\left(\mu_1\left(x_1\right), x_2\right) \ , \\
0 &= (-1)^{|x_2||x_3|+1}\mu_2\left(\mu_2\left(x_1, x_3\right), x_2\right) + (-1)^{|x_1|(|x_2|+|x_3|)}\mu_2\left(\mu_2\left(x_2, x_3\right), x_1\right) \\
&\quad + (-1)^{|x_1||x_2|+1}\mu_3\left(\mu_1\left(x_2\right), x_1, x_3\right) + (-1)^{(|x_1|+|x_2|)|x_3|}\mu_3\left(\mu_1\left(x_3\right), x_1, x_2\right) \\
&\quad + \mu_1\left(\mu_3\left(x_1, x_2, x_3\right)\right) + \mu_2\left(\mu_2\left(x_1, x_2\right), x_3\right) + \mu_3\left(\mu_1\left(x_1\right), x_2, x_3\right) \ ,
\end{aligned} \tag{2.6}$$

with x_1, x_2, x_3 being any combination of elements in either $\mathfrak{g}_0$ or $\mathfrak{g}_1$. These relations show that μ_1 is a graded differential compatible with μ_2, and, furthermore, μ_2 is a generalization of a Lie bracket with the violation of the Jacobi identity controlled by μ_3.

As such, it is not surprising that 2-term L_∞-algebras are equivalent to Lie 2-algebras, which was shown in [6]. Indeed, given a Lie 2-algebra L a corresponding 2-term L_∞-algebra $\mathfrak{g}$ is constructed using

$$\begin{aligned}
\mathfrak{g}_0 &= L_0 \ , \\
\mathfrak{g}_1 &= \ker(s) \subset L_1 \ ,
\end{aligned} \tag{2.7}$$

where L_0 and L_1 are, respectively, the objects and morphisms of the underlying 2-vector space of L and $s : L_1 \to L_0$ is its source map. The maps μ_i for $\mathfrak{g}$ are defined to be

$$
\begin{aligned}
\mu_1(h) &= t(h), && h \in \mathfrak{g}_1 \subset L_1 \ , \\
\mu_2(x,y) &= [x,y], && x,y \in \mathfrak{g}_0 = L_0 \ , \\
\mu_2(x,h) &= [i(x),h], && h \in \mathfrak{g}_1 \subset L_1, x \in \mathfrak{g}_0 = L_0 \ , \\
\mu_3(x,y,z) &= J_{x,y,z}, && x,y,z, \in \mathfrak{g}_0 = L_0 \ ,
\end{aligned}
\tag{2.8}
$$

where $i : L_0 \to L_1$ is the identity-assigning morphism, $t : L_1 \to L_0$ the target map and $J_{x,y,z}$ the Jacobiator of L. These maps then satisfy the homotopy Jacobi relations (2.2).

Conversely, given a 2-term L_∞-algebra $\mathfrak{g}$ we construct a Lie 2-algebra L by defining the underlying 2-vector space to be

$$
\begin{aligned}
L_0 &= \mathfrak{g}_0 \ , \\
L_1 &= \mathfrak{g}_0 \oplus \mathfrak{g}_1 \ .
\end{aligned}
\tag{2.9}
$$

Writing $x \in L_0$, $f = (f_0, f_1) \in L_1$ and using the maps μ_i of $\mathfrak{g}$ we can then define the source, target and identity-assigning morphisms to be

$$
\begin{aligned}
&s(f) = s(f_0,f_1) = f_0 \ , \\
&t(f) = t(f_0,f_1) = f_0 + \mu_1(f_1) \ , \\
&i(x) = (x,0) \ .
\end{aligned}
\tag{2.10}
$$

Furthermore, we can define the bracket functor on objects and morphisms as follows

$$
\begin{aligned}
&[x,y] = \mu_2(x,y) \ , \\
&[f,g] = (\mu_2(f_0,g_0), \mu_2(f_0,g_1) - \mu_2(g_0,f_1)) \ ,
\end{aligned}
\tag{2.11}
$$

where $x, y \in L_0$ and $f, g \in L_1$. Lastly, the Jacobiator can be defined as

$$
J_{x,y,z} = (\mu_2(\mu_2(x,y),z), \mu_3(x,y,z)) \ , \tag{2.12}
$$

which, by construction, satisfies the Jacobiator identity yielding a semi-strict Lie 2-algebra.

Note from (2.10), that in the identification of a semi-strict Lie 2-algebra with its corresponding L_∞-algebra we have that $\mu_1(f_1) = t(f) - s(f)$. Therefore, if the category underlying the Lie 2-algebra is skeletal, which for 2-vector spaces implies that source and target maps are always equal[1], we conclude that the corresponding L_∞-algebra has vanishing μ_1. As such, we call 2-term L_∞-algebras, for which μ_1 vanishes, **skeletal**. It can be shown that any Lie 2-algebra is equivalent to a skeletal one, see [6].

In similar fashion, a strict Lie 2-algebra is a a semi-strict Lie 2-algebra, for which the Jacobi identity holds identically, i.e. the Jacobiator is just the identity. Comparing with (2.12), this corresponds to the 2-term L_∞-algebra having vanishing μ_3 and we call such an L_∞-algebra **strict**. These strict 2-term L_∞-algebras can also be identified with the more familiar concept of a differential crossed module:

Definition 2.6 (Differential crossed module)

*A **differential crossed module** consists of two Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ together with a homomorphism $t : \mathfrak{h} \to \mathfrak{g}$ and an action $\alpha : \mathfrak{g} \to \mathfrak{Der}(\mathfrak{h})$ of $\mathfrak{g}$ on $\mathfrak{h}$ by derivations satisfying*

$$\begin{aligned} t(\alpha(x)(y)) &= [x, t(y)] \ , \\ \alpha(t(y_1))(y_2) &= [y_1, y_2] \ , \end{aligned} \tag{2.13}$$

where $x \in \mathfrak{g}$ and $y_1, y_2 \in \mathfrak{h}$.

This is straightforwardly the same as a strict 2-term L_∞-algebra, where the homomorphism t is identified with $\mu_1 : \mathfrak{h} = \mathfrak{g}_1 \to \mathfrak{g} = \mathfrak{g}_0$ and the action α yields the mixed $\mu_2 : \mathfrak{g}_0 \wedge \mathfrak{g}_1 \to \mathfrak{g}_1$.

As 2-term L_∞-algebras are equivalent to Lie 2-algebras and are more convenient to work with, we will use these exclusively in the remainder of the thesis. Furthermore, they offer a straightforward generalization to n-term L_∞-algebras for arbitrary n, which will feature prominently throughout the following.

[1] In general, a skeletal category is a category for which all isomorphic objects are necessarily identical. For a 2-vector space all maps have an inverse as composition is defined as the vector space addition on the arrow part of morphisms and therefore a skeletal 2-vector space must have only maps for which source and target maps agree.

2.2 L_∞-algebras as co-algebras and Q-manifolds

There are alternative ways of describing L_∞-algebras, as opposed to Definition 2.5 in terms of multi-brackets. One elegant way is to describe L_∞-algebras and their homotopy Jacobi relations in terms of graded co-algebras and co-derivations, which is due to [58], see also [59, 64, 66]. Let us first define these notions to fix notation:

Definition 2.7 (Graded co-algebra)
*Let $\mathbb{K}$ be a field. A **graded co-algebra** is a graded $\mathbb{K}$-vector space C equipped with a linear, degree zero map, i.e. a co-multiplication $\Delta : C \to C \otimes C$, satisfying*

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta \ . \tag{2.14}$$

*Furthermore, the graded co-algebra is called **co-commutative** if*

$$\tau \circ \Delta = \Delta \ , \tag{2.15}$$

where $\tau : C \otimes C \to C \otimes C$ is the involution $x_1 \otimes x_2 \to (-1)^{|x_1||x_2|} x_2 \otimes x_1$ and $|\cdot|$ denotes the degree.

A graded vector space V naturally gives rise to such a co-commutative graded co-algebra $\vee^\bullet(V)$. To see this, consider the tensor space[2] $T^\bullet(V) = \bigoplus_{n\geq 1} V^{\otimes n}$ with the deconcatenation product

$$x_1 \otimes \cdots \otimes x_n \mapsto \sum_{i+j=n} x_1 \otimes \cdots \otimes x_i \bigotimes x_{i+1} \otimes \cdots \otimes x_n \ . \tag{2.16}$$

The graded co-algebra $\vee^\bullet(V)$ is then the subalgebra spanned by tensors $x_1 \vee \cdots \vee x_n$ that are fixed under the permutation action of S_n on the components $V^{\otimes n}$. This forms a co-commutative graded co-algebra with the co-multiplication given by

$$\Delta(x_1 \vee \cdots \vee x_n) = \sum_{i+j=n} \sum_{\sigma \in S_{i|j}} \epsilon(\sigma; x_1, \ldots, x_n)(x_{\sigma(1)} \vee \cdots \vee x_{\sigma(i)}) \otimes (x_{\sigma(i+1)} \vee \cdots \vee x_{\sigma(n)}) \ , \tag{2.17}$$

where $S_{i|j}$ again denotes the set of (i,j)-unshuffles and $\epsilon(\sigma; x_1, \ldots, x_n)$ is the **graded**

[2]The tensor space usually includes the ground field $\mathbb{K} =: V^0$ in the sum. We suppress this as we want to consider the co-algebra $\vee^\bullet(V)$ without the ground field — often referred to as the reduced graded symmetric co-algebra.

symmetric Koszul sign defined via the products

$$x_1 \vee \cdots \vee x_n = \epsilon(\sigma; x_1, \ldots, x_n) x_{\sigma(1)} \vee \cdots \vee x_{\sigma(n)} \ , \tag{2.18}$$

where now transposing two odd degree elements acquires a minus sign. This is related to the graded antisymmetric Koszul sign via the equation $\epsilon(\sigma) = \mathrm{sgn}(\sigma)\chi(\sigma)$, compare Definition 2.5.

Definition 2.8 (Morphism of graded co-algebras)
A ***morphism of graded co-algebras*** *between co-algebras* (C, Δ) *and* (C', Δ') *is a degree* 0*, linear map* $\Psi : C \to C'$ *satisfying*

$$(\Psi \otimes \Psi) \circ \Delta = \Delta' \circ \Psi \ . \tag{2.19}$$

Consider again the example of the graded symmetric vector space discussed above. Crucially, a morphism $\Psi : \vee^\bullet(V) \to \vee^\bullet(V')$ is, in fact, entirely determined by its image on V' itself, i.e. by

$$\Psi^1 : \vee^\bullet(V) \to V', \quad \Psi^1 = \mathrm{pr}|_{V'} \circ \Psi \ , \tag{2.20}$$

see e.g. [66, Appendix A], [67, Lemma 22.1] or Appendix A.2 for more details and explicit formulae. An analogous fact for co-derivations is what allows to build a co-derivation out of the multi-linear maps of an L_∞-algebra in the following.

Definition 2.9 (Co-derviation)
A ***co-derivation*** *on a co-algebra* (C, Δ) *is a degree* -1*, linear map* $\mathcal{D} : C \to C$ *satisfying*

$$\Delta \circ \mathcal{D} = (\mathcal{D} \otimes \mathrm{id} + \mathrm{id} \otimes \mathcal{D}) \circ \Delta \ . \tag{2.21}$$

In the case of a co-derivation $\mathcal{D}$ on $\vee^\bullet(V)$ we, similarly to the case of morphisms above, have that $\mathcal{D}$ is uniquely determined by its image projected onto V, i.e. by

$$\mathcal{D}^1 : \vee^\bullet(V) \to V, \quad \mathcal{D}^1 = \mathrm{pr}|_V \circ \mathcal{D} \ . \tag{2.22}$$

The full co-derivation can then be recovered via the formula

$$\mathcal{D}(x_1 \vee \cdots \vee x_n) = \sum_{i+j=n} \sum_{\sigma \in S_{i|j}} \epsilon(\sigma; x_1, \ldots, x_n) \mathcal{D}^1(x_{\sigma(1)} \vee \cdots \vee x_{\sigma(i)}) \vee x_{\sigma(i+1)} \vee \cdots \vee x_{\sigma(n)} \; , \tag{2.23}$$

where $S_{i|j}$ again denotes the (i, j)-unshuffles.

With this, we can now see how an n-term L_∞-algebra can be encoded using graded symmetric co-algebras and corresponding co-derivations: Given an n-term L_∞-algebra $\mathfrak{g}$, consider the space $\mathfrak{g}[1]$ with the generators' degree shifted by one. Consequently, the maps μ_i are shifted in degree from $i-2$ to $i-2-i+1=-1$. This allows, using (2.23), to define a consistent co-derivation on $\vee^\bullet(\mathfrak{g}[1])$ given by

$$\mathcal{D} : \vee^\bullet(\mathfrak{g}[1]) \to \vee^\bullet(\mathfrak{g}[1]), \quad \mathcal{D}^1 = \sum_{i=1}^{n+1} (-1)^{\frac{i(i-1)}{2}} s \circ \mu_i \circ (s^{-1})^{\otimes i} \; . \tag{2.24}$$

Here, $s : \mathfrak{g} \to \mathfrak{g}[1]$ is the degree-shift made explicit and the minus signs are inserted to make connection with the usual conventions. Given this, the condition $\mathcal{D}^2 = 0$ directly translate to the homotopy Jacobi relations (2.2) motivating the following definition:

Definition 2.10 (L_∞-algebra as graded co-algebra)
*An **L_∞-algebra** is a pair $(\mathfrak{g}, \mathcal{D})$ where $\mathfrak{g}$ is a graded vector space and $\mathcal{D}$ is a co-derivation on $\vee^\bullet(\mathfrak{g}[1])$, that squares to zero, i.e. $\mathcal{D}^2 = 0$. An **n-term L_∞-algebra** is such a pair, where $\mathfrak{g}$ is non-trivial only in degrees $0, \ldots, n-1$.*

This viewpoint on L_∞-algebras is due to [58]. In fact, every such co-differential comes about this way, see [59].

An illuminating example is given by ordinary Lie algebras $\mathfrak{g}$ themselves: let t_α denote the generator of degree 1 in $\mathfrak{g}[1]$. Then, the co-differential is given by

$$\mathcal{D}^1 : \vee^2(\mathfrak{g}[1]) \to \mathfrak{g}[1], \quad \mathcal{D}^1(t_\alpha \vee t_\beta) = f_{\alpha\beta}^\gamma t_\gamma \; , \tag{2.25}$$

where $f_{\alpha\beta}^\gamma$ are the structure constants of $\mathfrak{g}$. Using the full co-derivation as defined in (2.23) we compute

$$\begin{aligned}\mathcal{D}^2(t_\alpha \vee t_\beta \vee t_\gamma) &= \mathcal{D}(f^\delta_{\alpha\beta} t_\delta \vee t_\gamma - f^\delta_{\alpha\gamma} t_\delta \vee t_\beta + f^\delta_{\beta\gamma} t_\delta \vee t_\alpha) \\ &= f^\delta_{\alpha\beta} f^\epsilon_{\delta\gamma} t_\epsilon - f^\delta_{\alpha\gamma} f^\epsilon_{\delta\beta} t_\epsilon + f^\delta_{\beta\gamma} f^\epsilon_{\delta\alpha} t_\epsilon \ .\end{aligned} \tag{2.26}$$

Therefore, the condition $\mathcal{D}^2 = 0$ just translates to $f^\delta_{[\alpha\beta} f^\epsilon_{\gamma]\delta} = 0$, which is just the Jacobi identity for $\mathfrak{g}$. To see that, in general, the condition $\mathcal{D}^2 = 0$ translates to the full relations in (2.2), see Appendix A.1.

A major advantage of this viewpoint is that it is now immediately clear what a morphism between arbitrary n-term L_∞-algebras should be.

Definition 2.11 (Morphism of L_∞-algebras in terms of co-algebras)
A ***morphism of L_∞-algebras*** *between L_∞-algebras $(\mathfrak{g}, \mathcal{D})$ and $(\mathfrak{g}', \mathcal{D}')$ is a morphism $\Psi : \vee^\bullet(\mathfrak{g}[1]) \to \vee^\bullet(\mathfrak{g}'[1])$ of the underlying differential graded co-algebra (Definition 2.8) that respects the co-differential, i.e. $\Psi \circ \mathcal{D} = \mathcal{D}' \circ \Psi$.*

Let us illustrate this in the example of 2-term L_∞-algebras: let t_α and b_a denote the generators of respective degree 1 and 2 in $(\mathfrak{g}, \mathcal{D})$ corresponding to the 2-term L_∞-algebra $\mathfrak{g}$. A generic co-differential $\mathcal{D}$ with generalized structure constants f is given on these generators by

$$\begin{aligned}\mathcal{D}^1(b_a) &= f^\alpha_a t_\alpha \ , \\ \mathcal{D}^1(t_\alpha \vee t_\beta) &= f^\gamma_{\alpha\beta} t_\gamma \ , \\ \mathcal{D}^1(t_\alpha \vee b_a) &= f^b_{\alpha a} b_b \ , \\ \mathcal{D}^1(t_\alpha \vee t_\beta \vee t_\gamma) &= f^a_{\alpha\beta\gamma} b_a \ ,\end{aligned} \tag{2.27}$$

where again (2.23) is used to construct the full co-differential $\mathcal{D}$. Analogously defining a 2-term L_∞-algebra $(\mathfrak{g}', \mathcal{D}')$ we can write down a generic morphism $\Psi : \vee^\bullet(\mathfrak{g}[1]) \to \vee^\bullet(\mathfrak{g}'[1])$, which is determined by its image Ψ^1 on $\mathfrak{g}'[1]$ given by

$$\begin{aligned}\Psi^1(t_\alpha) &= \Psi^\beta_\alpha t'_\beta \ , \\ \Psi^1(b_a) &= \Psi^b_a b'_b \ ,\end{aligned} \tag{2.28}$$

$$\Psi^1(t_\alpha \vee t_\beta) = \Psi^a_{\alpha\beta} b'_a \ .$$

The condition $\Psi \circ \mathcal{D} = \mathcal{D}' \circ \Psi$ then leads to

$$\begin{aligned}
0 &= \Psi^\gamma_\beta f^\beta_\alpha - f'^\gamma_\beta \Psi^\beta_\alpha \ , \\
0 &= \Psi^\beta_\alpha f^\alpha_a - f'^\beta_b \Psi^b_a \ , \\
0 &= \Psi^\epsilon_\gamma f^\gamma_{\alpha\beta} - f'^\epsilon_a \Psi^a_{\alpha\beta} - f'^\epsilon_{\gamma\delta} \Psi^\gamma_\alpha \Psi^\delta_\beta \ , \\
0 &= \Psi^c_b f^b_{\alpha a} - \Psi^c_{\alpha\beta} f^\beta_a - f'^c_{\beta b} \Psi^\beta_\alpha \Psi^b_a \ , \\
0 &= \Psi^b_a f^a_{\alpha\beta\gamma} + \Psi^b_{\delta\gamma} f^\delta_{\alpha\beta} - \Psi^b_{\delta\beta} f^\delta_{\alpha\gamma} + \Psi^b_{\delta\alpha} f^\delta_{\beta\gamma} \\
&\quad - f'^b_{\delta\epsilon\zeta} \Psi^\delta_\alpha \Psi^\epsilon_\beta \Psi^\zeta_\gamma - f'^b_{\delta a} \Psi^\delta_\alpha \Psi^a_{\beta\gamma} + f'^b_{\delta a} \Psi^\delta_\beta \Psi^a_{\alpha\gamma} - f'^b_{\delta a} \Psi^\delta_\gamma \Psi^a_{\alpha\beta} \ .
\end{aligned} \tag{2.29}$$

When translated back to the multi-bracket point of view this precisely agrees with the more familiar expressions for a morphism between 2-term L_∞-algebras as given in [6]. The notion of morphisms given here however now readily extends to an arbitrary n-term L_∞-algebra. For a more precise derivation and explicit formulae for such morphisms, see Appendix A.2.

Finally, a third viewpoint on L_∞-algebras is given by the dual of the view above, i.e. by differential graded algebras. This can also be understood as special cases of Q-manifolds, more familiar to physicists from e.g. BRST quantization, see for instance [68].

Definition 2.12 (Q-manifold)
*A **Q-manifold** is a graded manifold, that is, a manifold M with an additional $\mathbb{Z}$-grading in the structure sheaf, together with a homological vector field Q, that is, a degree 1 vector field that squares to zero: $Q^2 = 0$.*

An easy way to think of Q-manifolds is as a tower of fibrations

$$\mathcal{M}_0 \longleftarrow \mathcal{M}_1 \longleftarrow \mathcal{M}_2 \longleftarrow \mathcal{M}_3 \longleftarrow \dots \ , \tag{2.30}$$

where $\mathcal{M}_0 := M$ is the manifold and $\mathcal{M}_i$ are vector bundles with fibre coordinates of degree i. These generate the graded structure sheaf, that is, the graded structure sheaf is generated by polynomials on those graded coordinates. For more details on this, see e.g. [68]. The algebra of these functions together with the vector field Q then form a differential graded algebra.

To illuminate the connection to L_∞-algebras consider an L_∞-algebra $\mathfrak{g}$ given by a co-algebra as in Definition 2.10. Assuming $\mathfrak{g}$ to be finite-dimensional[3], we can then dualize this concept. The graded co-algebra $\vee^\bullet(\mathfrak{g}[1])$ becomes the graded algebra[4] $\wedge^\bullet(\mathfrak{g}^*[1])$, and the co-derivation $\mathcal{D}$ induces a derivation $Q : \wedge^\bullet(\mathfrak{g}^*[1]) \to \wedge^\bullet(\mathfrak{g}^*[1])$, which acts on the graded algebra $\wedge^\bullet(\mathfrak{g}^*[1])$. In the process of dualizing, the degrees in $\mathfrak{g}^*[1]$ would become negative; however, for convenience, we insert a minus sign, such that all coordinates have positive degrees[5]. This implies that now Q is of degree 1 instead of -1 and as such this fits into the framework of Q-manifolds: Indeed, the tower of fibration here is given by

$$\mathfrak{g}^*[1] = (* \leftarrow \mathfrak{g}_0^*[1] \leftarrow \cdots \leftarrow \mathfrak{g}_{n-1}^*[1]) \ , \tag{2.31}$$

i.e. for an L_∞-algebra the manifold M is just the point. The more general case, where M is a non-trivial manifold is called an L_∞-algebroid, which, however, will not be relevant in the remainder of the thesis. The resulting differential graded algebra $(\wedge^\bullet(\mathfrak{g}^*[1]), Q)$ is also known as the Chevalley–Eilenberg algebra of $\mathfrak{g}$ and will be discussed in more detail in Section 2.3.

Another important example for Q-manifolds is that of the shifted tangent bundle $T[1]M$ of a manifold M, where we have coordinates x^μ of degree 0 on the base and coordinates ξ^μ of degree 1 on the fiber. We can endow $T[1]M$ with the vector field $Q = \xi^\mu \frac{\partial}{\partial x^\mu}$, which naturally squares to zero. Then, the algebra of functions on $T[1]M$ can be identified with the algebra $\Omega^\bullet(M)$ of differential forms. Moreover, under this identification the vector field Q becomes the usual de Rham differential d, and, thus, the de Rham complex $(\Omega^\bullet(M), \mathrm{d})$ forms a Q-manifold.

[3] This assumption will not always be satisfied in the examples considered in this thesis. In these cases, one can either work with the co-algebra picture or introduce additional dual coordinates that yield a sufficient basis for the relevant calculations.

[4] We use the symbols $\wedge$ and $\vee$ to emphasize the algebra and co-algebra nature. Both of these follow graded symmetric conventions.

[5] We adopt this convention throughout the thesis, so that in all viewpoints the coordinates have non-negative degrees.

A similar, but more involved example, which will be of relevance in Section 3.6, is given by $\mathcal{V}_2 \coloneqq T^*[2]T[1]M$. As the functor T^* gives extra coordinates with opposite degree, we have local coordinates $(x^\mu, \xi^\mu, \xi_\mu, p_\mu)$ of degree 0, 1, 1, and 2, respectively. A canonical choice of homological vector field is now

$$Q_{\mathcal{V}_2} = \xi^\mu \frac{\partial}{\partial x^\mu} + p_\mu \frac{\partial}{\partial \xi_\mu} \ , \tag{2.32}$$

This example is part of a larger class of Q-manifolds given by $\mathcal{V}_n \coloneqq T^*[n]T[1]M$ containing the Vinogradov algebroids $TM \oplus \wedge^{n-1}T^*M$. For more details, see e.g. [64] or [68].

2.3 Chevalley–Eilenberg algebra, Weil algebra and invariant polynomials

As discussed in the previous section, one can also view L_∞-algebras from the point of view of differential graded algebras. The definitions of such graded algebras, their morphisms and differentials are the duals of Definitions 2.7, 2.8 and 2.9. As these definitions are more familiar we will neglect to spell them out in detail and just note that morphisms and differentials are entirely defined by their action on generators. This is the dual notion to co-algebra morphisms and co-derivations being defined entirely by their projected images, see (2.20) and (2.25).

For L_∞-algebras we give the dual of Definition 2.10, which is known as the Chevalley–Eilenberg algebra of an L_∞-algebra:

Definition 2.13 (Chevalley–Eilenberg algebra)
The ***Chevalley–Eilenberg algebra*** $\mathsf{CE}(\mathfrak{g})$ *of an L_∞-algebra $\mathfrak{g}$ is the differential graded algebra based on* $\wedge^\bullet(\mathfrak{g}^*[1])$ *together with a differential* $Q : \wedge^\bullet(\mathfrak{g}^*[1]) \to \wedge^\bullet(\mathfrak{g}^*[1])$ *of degree 1, which squares to zero, that is,* $Q^2 = 0$. *For an n-term L_∞-algebra,* $\mathfrak{g}^*[1]$ *is concentrated in degrees* $1, \ldots, n$.

Analogously to the co-algebra viewpoint, the condition $Q^2 = 0$ corresponds to the homotopy Jacobi relations (2.2). For clarity, let us repeat the example of an ordinary Lie algebra $\mathfrak{g}$ in this language: Let t^α denote the generator of degree 1 in

$\mathfrak{g}^*[1]$. The differential Q is then given by

$$Qt^\alpha = -\tfrac{1}{2} f^\alpha_{\beta\gamma} t^\beta \wedge t^\gamma \ , \tag{2.33}$$

where $f^\alpha_{\beta\gamma}$ are the structure constants of $\mathfrak{g}$. This leads to

$$Q^2 t^\alpha = Q(-\tfrac{1}{2} f^\alpha_{\beta\gamma} t^\beta \wedge t^\gamma) = f^\alpha_{\beta\gamma} t^\beta \wedge Q t^\gamma = -\tfrac{1}{2} f^\alpha_{\beta\gamma} f^\gamma_{\delta\epsilon} t^\beta \wedge t^\delta \wedge t^\epsilon = 0 \ . \tag{2.34}$$

As the generators t^α are of degree 1 and therefore anti-commute, we can see that this is just the Jacobi identity $f^\alpha_{\gamma[\beta} f^\gamma_{\ \delta\epsilon]} = 0$ of the Lie algebra $\mathfrak{g}$. Again, see Appendix A.1 for the general case.

Just as in the co-algebra picture, it is immediately clear what a morphism for L_∞-algebras should be in this picture.

Definition 2.14 (Morphism of L_∞-algebras in terms of algebras)
*A **morphism of L_∞-algebras** between L_∞-algebras $(\mathfrak{g}, Q)$ and $(\mathfrak{g}', Q')$ is a morphism $\Phi : \wedge^\bullet(\mathfrak{g}^*[1]) \to \wedge^\bullet(\mathfrak{g}'^*[1])$ of degree 0, that preserves the wedge product and respects the differential, i.e. $\Phi \circ Q = Q' \circ \Phi$.*

This is dual to Definition 2.11 and, again, reproduces the morphisms for 2-term L_∞-algebras as given in [6], while giving a straightforward generalization to arbitrary n-term L_∞-algebras. For explicit formulae see Appendix A.2. Note that we only have to define such morphisms on the generators in $\mathfrak{g}^*[1]$. This is just the dual statement to equation (2.20).

As will become clear in Section 3.1, it is necessary to introduce Weil algebras, which, in essence, are a doubling of the Chevalley–Eilenberg algebra, in order to include the notion of curvatures in higher gauge theory. The Weil algebra is analogous to the Chevalley–Eilenberg algebra, where we insert another shifted copy of $\mathfrak{g}^*$. More specifically, we have the following definition.

Definition 2.15 (Weil algebra)
*The **Weil algebra** $\mathsf{W}(\mathfrak{g})$ of an L_∞-algebra $\mathfrak{g}$ is the differential graded algebra defined on the space $\wedge^\bullet(\mathfrak{g}^*[1] \oplus \mathfrak{g}^*[2])$ together with the differential Q given by*

$$Q_{\mathsf{W}}|_{\wedge^\bullet(\mathfrak{g}^*[1])} = Q_{\mathsf{CE}} + \sigma \ , \tag{2.35}$$

which is then uniquely extended to shifted generators by the requirement that Q_{W} squares to zero. Here, Q_{CE} denotes the differential of the Chevalley–Eilenberg algebra of $\mathfrak{g}$ and $\sigma : \mathfrak{g}^*[1] \to \mathfrak{g}^*[2]$ is the shift isomorphism, that identifies the generators in $\mathfrak{g}^*[1]$ with the corresponding generators in $\mathfrak{g}^*[2]$.

Note that the differential Q_{W} as given in (2.35) together with the requirement $Q_{\mathsf{W}}^2 = 0$ is sufficient to uniquely define it on shifted generators. To see this, let a be an element of $\wedge^\bullet(\mathfrak{g}^*[1])$. Then,

$$Q_{\mathsf{W}}^2 a = Q_{\mathsf{W}}(Q_{\mathsf{CE}}a + \sigma a) = \sigma Q_{\mathsf{CE}}a + Q_{\mathsf{W}}\sigma a \ , \tag{2.36}$$

where we used the fact, that, by definition, $Q_{\mathsf{CE}}a$ entirely lies inside $\wedge^\bullet(\mathfrak{g}^*[1])$. This implies,

$$Q_{\mathsf{W}}\sigma a = -\sigma Q_{\mathsf{CE}}a \ , \tag{2.37}$$

which uniquely defines Q_{W} on $\wedge^\bullet(\mathfrak{g}^*[1] \oplus \mathfrak{g}^*[2])$.

The Weil algebra is constructed in such a way that the step from Chevalley–Eilenberg algebra to Weil algebra is functorial in the sense that a morphism of Weil algebras is entirely and uniquely determined by the underlying morphism of Chevalley–Eilenberg algebras. Given a morphism $\Phi : \mathsf{CE}(\mathfrak{g}) \to \mathsf{CE}(\mathfrak{h})$, defined on generators $a \in \mathfrak{g}^*[1]$ as $a \mapsto \Phi(a)$, we can extend it to a morphism $\hat{\Phi} : \mathsf{W}(\mathfrak{g}) \to \mathsf{W}(\mathfrak{h})$ using $\sigma a \mapsto \sigma\Phi(a)$. We can directly check that this is still a morphism as in Definition 2.14 by checking the commutativity of the relevant squares:

$$\begin{tikzcd} a \arrow[r, "Q_{\mathsf{W}(\mathfrak{g})}"] \arrow[d, "\hat{\Phi}"] & Q_{\mathsf{CE}(\mathfrak{g})}a + \sigma a \arrow[d, "\hat{\Phi}"] \\ \Phi(a) \arrow[r, "Q_{\mathsf{W}(\mathfrak{g})}"] & Q_{\mathsf{CE}(\mathfrak{h})}\Phi(a) + \sigma\Phi(a) \end{tikzcd} \qquad \begin{tikzcd} \sigma a \arrow[r, "Q_{\mathsf{W}(\mathfrak{g})}"] \arrow[d, "\hat{\Phi}"] & -\sigma Q_{\mathsf{CE}(\mathfrak{g})}a \arrow[d, "\hat{\Phi}"] \\ \sigma\Phi(a) \arrow[r, "Q_{\mathsf{W}(\mathfrak{h})}"] & -\sigma Q_{\mathsf{CE}(\mathfrak{h})}\Phi(a) \ . \end{tikzcd} \tag{2.38}$$

Note, that the Weil algebra $\mathsf{W}(\mathfrak{g})$ can be seen as the Chevalley–Eilenberg algebra arising from some other L_∞-algebra, which we denote by $\mathsf{inn}(\mathfrak{g})$. For an n-term L_∞-algebra $\mathfrak{g}$, this $(n+1)$-term L_∞-algebra $\mathsf{inn}(\mathfrak{g})$ arises when adding a shifted copy $\mathfrak{g}[1]$ together with all the maps resulting from the shift isomorphism σ. For an ordinary Lie algebra $\mathfrak{g}$ this is the strict 2-term L_∞-algebra with μ_1 being the identity and the mixed μ_2 being the adjoint action of $\mathfrak{g}$ on $\mathfrak{g}[1]$. This is the Lie 2-algebra that corresponds to the 2-group of inner automorphisms discussed in [69].

Furthermore, the Weil algebra is naturally isomorphic to the free algebra, which is the algebra that corresponds to a Chevalley–Eilenberg algebra with vanishing differential.

Definition 2.16 (Free algebra)
*The **free algebra*** $\mathsf{F}(\mathfrak{g})$ *of an* L_∞*-algebra* $\mathfrak{g}$ *is the differential graded algebra defined on the space* $\wedge^\bullet(\mathfrak{g}^*[1] \oplus \mathfrak{g}^*[2])$ *together with the differential*

$$Q_{\mathsf{F}}|_{\mathfrak{g}^*[1]} = \sigma \qquad \text{and} \qquad Q_{\mathsf{F}}|_{\mathfrak{g}^*[2]} = 0 \ , \tag{2.39}$$

where $\sigma : \mathfrak{g}^*[1] \to \mathfrak{g}^*[2]$ *is the shift isomorphism.*

To see that $\mathsf{W}(\mathfrak{g})$ and $\mathsf{F}(\mathfrak{g})$ are isomorphic, consider the morphisms

$$\begin{aligned} \Phi : \mathsf{F}(\mathfrak{g}) \to \mathsf{W}(\mathfrak{g}), \quad & a \mapsto a \ , \\ & \sigma a \mapsto Q_{\mathsf{W}} a \ , \\ \Phi^{-1} : \mathsf{W}(\mathfrak{g}) \to \mathsf{F}(\mathfrak{g}), \quad & a \mapsto a \ , \\ & \sigma a \mapsto \sigma a - Q_{\mathsf{CE}} a \ , \end{aligned} \tag{2.40}$$

where a denotes the generators in $\mathfrak{g}^*[1]$ and σa the corresponding elements in $\mathfrak{g}^*[2]$. The compositions $\Phi \circ \Phi^{-1}$ and $\Phi^{-1} \circ \Phi$ directly yield the identity and we just have to check that the differential is respected, which can be seen in the commutativity of the squares

$$\begin{tikzcd} a \arrow[r, "Q_{\mathsf{F}}"] \arrow[d, "\Phi"] & \sigma a \arrow[d, "\Phi"] \\ a \arrow[r, "Q_{\mathsf{W}}"] & Q_{\mathsf{W}} a \end{tikzcd} \qquad \begin{tikzcd} \sigma a \arrow[r, "Q_{\mathsf{F}}"] \arrow[d, "\Phi"] & 0 \arrow[d, "\Phi"] \\ Q_{\mathsf{W}} a \arrow[r, "Q_{\mathsf{W}}"] & 0 \end{tikzcd} \tag{2.41}$$

and

$$\begin{tikzcd} a \arrow[r, "Q_{\mathsf{W}}"] \arrow[d, "\Phi^{-1}"] & Q_{\mathsf{W}} a \arrow[d, "\Phi^{-1}"] \\ a \arrow[r, "Q_{\mathsf{F}}"] & Q_{\mathsf{CE}} a + \sigma a - Q_{\mathsf{CE}} a \end{tikzcd} \qquad \begin{tikzcd} \sigma a \arrow[r, "Q_{\mathsf{W}}"] \arrow[d, "\Phi^{-1}"] & -\sigma Q_{\mathsf{CE}} a \arrow[d, "\Phi^{-1}"] \\ \sigma a - Q_{\mathsf{CE}} a \arrow[r, "Q_{\mathsf{F}}"] & -\sigma Q_{\mathsf{CE}} a \ . \end{tikzcd} \tag{2.42}$$

This isomorphism will be important in defining 2-morphisms in Section 2.4.

The dual of the inclusion map canonically projects the Weil algebra $\mathsf{W}(\mathfrak{g})$ onto the Chevalley–Eilenberg algebra $\mathsf{CE}(\mathfrak{g})$, i.e.

$$
\begin{aligned}
i^*|_{\mathfrak{g}^*[1]} &= \mathrm{id} \;, \\
i^*|_{\mathfrak{g}^*[2]} &= 0 \;,
\end{aligned}
\tag{2.43}
$$

so that i^* respects the differentials and we have a morphism of differential graded algebras,

$$
\mathsf{CE}(\mathfrak{g}) \xleftarrow{i^*} \mathsf{W}(\mathfrak{g}) \;. \tag{2.44}
$$

A third element, that extends this sequence and is of importance to our discussion, is that of the invariant polynomials of an L_∞-algebra $\mathfrak{g}$. Such an invariant polynomial is an element $P \in \mathsf{W}(\mathfrak{g})|_{\wedge^\bullet(\mathfrak{g}^*[2])}$, which sits entirely in the shifted copy $\mathfrak{g}^*[2]$ inside the Weil algebra and is additionally closed under Q_{W}. We are interested in horizontal equivalence classes of such invariant polynomials leading to the following definition:

Definition 2.17 (Algebra of invariant polynomials)
*An **invariant polynomial** P of an L_∞-algebra $\mathfrak{g}$ is an element $P \in \mathsf{W}(\mathfrak{g})|_{\wedge^\bullet(\mathfrak{g}^*[2])}$, such that $Q_{\mathsf{W}} P = 0$. Two such invariant polynomials P_1 and P_2 are **horizontally equivalent**, $P_1 \sim P_2$, if there exists an element $\tau \in \ker(i^*)$ such that $P_1 = P_2 + Q_{\mathsf{W}}\tau$. The **algebra of invariant polynomials** $\mathsf{inv}(\mathfrak{g})$ of $\mathfrak{g}$ is the differential graded algebra of horizontal equivalence classes of invariant polynomials on $\mathfrak{g}$.*

More generally, one can consider invariant polynomials whose image under Q_{W} is not closed but lies entirely in $\wedge^\bullet(\mathfrak{g}^*[2])$, cf. [14]. Here, we however focus on closed invariant polynomials.

To connect this to the ordinary notion of invariant polynomials, consider the case of an ordinary Lie algebra $\mathfrak{g}$. Writing t^α and $r^\alpha = \sigma t^\alpha$ for the generators of $\mathsf{W}(\mathfrak{g})$ in degree 1 and 2, respectively, the differential for the Weil algebra is given by

$$
Q_{\mathsf{W}} t^\alpha = -\tfrac{1}{2} f^\alpha_{\beta\gamma} t^\beta \wedge t^\gamma + r^\alpha \qquad \text{and} \qquad Q_{\mathsf{W}} r^\alpha = -f^\alpha_{\beta\gamma} t^\beta \wedge r^\gamma \;, \tag{2.45}
$$

where, again, $f^\alpha_{\beta\gamma}$ are the structure constants of $\mathfrak{g}$. In these coordinates, an invariant polynomial is then given by

$$P = P_{(\alpha_1,\dots,\alpha_n)} r^{\alpha_1} \dots r^{\alpha_n} \in \wedge^\bullet(\mathfrak{g}^*[2]) \;, \tag{2.46}$$

where the brackets denote symmetrization. Here, P being Q_{W}-closed is the same as the condition that $Q_{\mathsf{W}}P$ lies in $\wedge^\bullet(\mathfrak{g}^*[2])$ and implies

$$\sum_{i=1}^{n} f^\gamma_{\beta\alpha_i} P_{\alpha_1,\dots,\hat{\alpha}_i,\gamma,\dots,\alpha_n} = 0 \;. \tag{2.47}$$

This is the more familiar expression of the invariance of P under the adjoint action of $\mathfrak{g}$, compare e.g. [70].

More generally, for an arbitrary n-term L_∞-algebra, let us consider the contraction derivation $\iota_X : \mathsf{W}(\mathfrak{g}) \to \mathsf{W}(\mathfrak{g})$, which is given by the contraction with an element $X \in \mathfrak{g}[1]$. Then, by definition, Lie derivatives $[Q_{\mathsf{W}}, \iota_X]$ vanish on any element in $\mathsf{inv}(\mathfrak{g})$, as these live in the kernel of all ι_X. In Chern–Weil theory, it is precisely these invariant polynomials applied to curvature forms that are identified with the characteristic classes of the group G integrating $\mathfrak{g}$.

Together with $\mathsf{CE}(\mathfrak{g})$ and $\mathsf{W}(\mathfrak{g})$ the algebra of invariant polynomials $\mathsf{inv}(\mathfrak{g})$ now forms the short sequence

$$\mathsf{CE}(\mathfrak{g}) \xleftarrow{\;i^*\;} \mathsf{W}(\mathfrak{g}) \xleftarrow{\;p^*\;} \mathsf{inv}(\mathfrak{g}) \;, \tag{2.48}$$

where i^* is the projection as before and p^* is the natural inclusion of $\mathsf{inv}(\mathfrak{g})$ in $\mathsf{W}(\mathfrak{g})$. This sequence will feature prominently in the definition of a $\mathfrak{g}$-connection object in Section 3.3 and subsequently in the discussion of the twisted string algebras in Section 3.5. Furthermore, we can now form the following definitions which will also be of relevance.

Definition 2.18 (Cocycles & $\mathfrak{g}$-transgression elements)
Let $\mathfrak{g}$ be an L_∞-algebra. An element $\mu \in \mathsf{CE}(\mathfrak{g})$ that closes under Q_{CE}, i.e. $Q_{\mathsf{CE}}\mu = 0$, is called an ***L_∞-algebra cocycle****. Given such a cocycle μ and an invariant*

polynomial $P \in \mathsf{inv}(\mathfrak{g})$ *we call an element* $\mathrm{cs} \in \mathsf{W}(\mathfrak{g})$ *that satisfies*

$$\begin{aligned} Q_{\mathsf{W}}\mathrm{cs} &= p^*(P) \ , \\ i^*(\mathrm{cs}) &= \mu \ , \end{aligned} \tag{2.49}$$

a $\mathfrak{g}$***-transgression element*** *or* ***Chern–Simons element*** *for* μ *and* P*.*

If such a transgression element exists for a given cocycle μ and invariant polynomial P, we say, that μ **transgresses to** P and P **suspends to** μ. Note that given this, μ is indeed a cocycle:

$$Q_{\mathsf{CE}}\mu \ = \ Q_{\mathsf{CE}}i^*(\mathrm{cs}) \ = \ i^*(Q_{\mathsf{W}}\mathrm{cs}) \ = \ i^*(p^*(P)) \ = \ 0 \ , \tag{2.50}$$

where we use the fact that i^* is a morphism of differential graded algebras and $\mathrm{im}(p^*) \subset \ker(i^*)$. The cohomology class of the cocycle μ is independent of the transgression element cs chosen. Indeed, considering $\mu' = \mu + Q_{\mathsf{CE}}a$ for some $a \in \mathsf{CE}(\mathfrak{g})$ we have $\mu' = i^*(\mathrm{cs} + Q_{\mathsf{W}}a)$ and $Q_{\mathsf{W}}(\mathrm{cs} + Q_{\mathsf{W}}a) = Q_{\mathsf{W}}\mathrm{cs} = p^*P$, so that μ' transgresses to the same invariant polynomial. Therefore, an invariant polynomial that suspends to a coboundary $\mu = Q_{\mathsf{CE}}a$ also suspends to 0.

This also behaves well with the concept of horizontal equivalence given in Definition 2.17: two invariant polynomials $P_1, P_2 \in \mathsf{inv}(\mathfrak{g})$ that suspend to the same cocycle μ are horizontally equivalent. This can be easily seen from

$$i^*(\mathrm{cs}_1 - \mathrm{cs}_2) \ = \ i^*(\mathrm{cs}_1) - i^*(\mathrm{cs}_2) \ = \ \mu - \mu \ = \ 0 \ , \tag{2.51}$$

so that $\mathrm{cs}_1 - \mathrm{cs}_2 \in \ker(i^*)$ and $P_1 - P_2 = Q_{\mathsf{W}}(\mathrm{cs}_1 - \mathrm{cs}_2)$.

Furthermore, there exists a transgression element for any invariant polynomial $P \in \mathsf{inv}(\mathfrak{g})$. This is due to the fact that all invariant polynomials are closed and the cohomology of the Weil algebra is trivial, which can be immediately seen from the isomorphism in (2.40). Then, any decomposable invariant polynomial, i.e. an invariant polynomial P that can be written as the product $P = P_1 \wedge P_2$ of two invariant polynomials of non-vanishing degree, suspends to 0: we have a transgessions element cs_1 for P_1. Additionally, as, by definition, P_2 is an element in the kernel of i^*, so is $\mathrm{cs}_1 \wedge P_2$ in $\ker(i^*)$ and we have $P = Q_{\mathsf{W}}(\mathrm{cs}_1 \wedge P_2)$. Therefore, the algebra of invariant polynomials $\mathsf{inv}(\mathfrak{g})$ only contains the indecomposable invariant polynomials.

2.4 2-morphisms for L_∞-algebras

So far we have discussed L_∞-algebras and a notion of morphism for these. We are additionally interested in a notion of 2-morphisms between such morphisms, which allows for a suitable notion of quasi-isomorphisms and equivalences of L_∞-algebras. A definition for these is in principle clear from category theory, see e.g. [71]. However, an explicit definition is not immediately available. If one is interested only in the equivalence of L_∞-algebras there is a convenient shortcut: a morphism of L_∞-algebras is a quasi-isomorphism if it induces an isomorphism on cohomology, cf. [72]. This still does not give explicit expressions for the morphisms going back and forth, but, fortunately, a suitably explicit definition of 2-morphisms was given in [14], which we shall recall here. This notion of 2-morphisms will yield an explicit form for equivalences between L_∞-algebras, i.e. a pair of morphisms Φ, Ψ whose compositions are connected to the identity morphism via a 2-morphism. This will be relevant in the discussion of twisted string algebras in Section 3.5.

The definition of 2-morphisms makes crucial use of the fact that the Weil algebra $\mathsf{W}(\mathfrak{g})$ is naturally isomorphic to the free algebra $\mathsf{F}(\mathfrak{g})$, see the discussion below Definition 2.16: given two morphisms Φ and Ψ between Chevalley–Eilenberg algebras $\mathsf{CE}(\mathfrak{g})$ and $\mathsf{CE}(\mathfrak{h})$ we first define a 2-morphism on the generators of the free algebra $\mathsf{F}(\mathfrak{g})$ isomorphic to $\mathsf{W}(\mathfrak{g})$, before extending it to the generators of $\mathsf{W}(\mathfrak{g})$.

Definition 2.19 (2-morphism for L_∞-algebras [14])
*Let Φ and Ψ be morphisms between L_∞-algebras $\mathfrak{g}$ and $\mathfrak{h}$, seen as the Chevalley–Eilenberg algebras $\mathsf{CE}(\mathfrak{g})$ and $\mathsf{CE}(\mathfrak{h})$. A **2-morphism** between Φ and Ψ, i.e.*

$$\begin{tikzcd} \mathsf{CE}(\mathfrak{h}) & \mathsf{CE}(\mathfrak{g}) \arrow[l, bend right, "\Phi"'] \arrow[l, bend left, "\Psi"] \arrow[l, phantom, "\Downarrow\eta"] \end{tikzcd} , \qquad (2.52)$$

is given by a degree -1 map

$$\eta : \mathfrak{g}^*[1] \oplus \mathfrak{g}^*[2] \to \mathsf{CE}(\mathfrak{h}) \qquad (2.53)$$

defined on the generators $\mathfrak{g}^*[1] \oplus \mathfrak{g}^*[2]$ *of the free algebra* $\mathsf{F}(\mathfrak{g})$ *such that on those generators* $\Phi - \Psi = [Q, \eta]$*. This morphism is extended to the full space* $\mathsf{F}(\mathfrak{g})$ *using the formula*

$$\begin{aligned} \eta : x_1 \wedge \cdots \wedge x_n \mapsto \tfrac{1}{n!} \sum_{\sigma \in S_n} \epsilon(\sigma) \sum_{k=1}^{n} (-1)^{\sum_{i=1}^{k-1} |x_{\sigma(i)}|} \Psi(x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(k-1)}) \\ \wedge\, \eta(x_{\sigma(k)}) \wedge \Phi(x_{\sigma(k+1)} \wedge \cdots \wedge x_{\sigma(n)}) \ , \end{aligned} \tag{2.54}$$

where $x_i \in \mathfrak{g}^*[1] \oplus \mathfrak{g}^*[2]$ *and* ϵ *is the graded symmetric Koszul sign. Additionally, we require that* η*, when viewed as a morphism out of* $\mathsf{W}(\mathfrak{g})$*, vanishes on the generators in the shifted copy inside* $\mathsf{W}(\mathfrak{g})$*, i.e. we have a diagram*

$$\begin{tikzcd} & \mathsf{CE}(\mathfrak{g}) \arrow[dl, bend right, "\Phi"'] & & \mathfrak{g}^*[2] \arrow[dl, hook] \\ \mathsf{CE}(\mathfrak{h}) & & \mathsf{W}(\mathfrak{g}) \arrow[ul, "i^*"'] \arrow[dl, "i^*"] \arrow[ll, Rightarrow, "\eta"] & \mathsf{F}(\mathfrak{g}) \arrow[l, "\cong"'] \ , \\ & \mathsf{CE}(\mathfrak{g}) \arrow[ul, bend left, "\Psi"] & & \end{tikzcd} \tag{2.55}$$

where η *vanishes along* $\mathfrak{g}^*[2] \hookrightarrow \mathsf{W}(\mathfrak{g})$*.*

Note, that formula (2.54) guarantees that $\Phi - \Psi = [Q, \eta]$ on the whole of $\mathsf{F}(\mathfrak{g})$. Comparing with the canonical isomorphism (2.41) we can see that the condition that η vanishes on the shifted generators explicitly implies that $\eta : \mathsf{F}(\mathfrak{g}) \to \mathsf{CE}(\mathfrak{h})$ defines a map on the generators a and $Q_{\mathsf{W}} a$ inside $\mathsf{W}(\mathfrak{g})$ in such a way that η vanishes on all $\sigma a \in \mathsf{W}(\mathfrak{g})$. It is important to employ formula (2.54) only on a and $Q_{\mathsf{W}} a$, i.e. those generators that come from the generators of $\mathsf{F}(\mathfrak{g})$, as otherwise ambiguities arise.

Furthermore, this definition is sufficient to cover morphisms between Weil algebras: recall, that $\mathsf{W}(\mathfrak{g})$ can be seen as the Chevalley–Eilenberg algebra $\mathsf{CE}(\mathsf{inn}(\mathfrak{g}))$ (see below equation (2.38)), which can straightforwardly be used to extend to the case of morphisms between Weil algebras.

It is instructive to spell out what this definition means explicitly. We will here deal with the simplest non-trivial case, which is that of 2-term L_∞-algebras $\mathfrak{g}$ and $\mathfrak{h}$. For formulae up to 3-term L_∞-algebras see Appendix A.3. Let $\Phi, \Psi : \mathsf{CE}(\mathfrak{g}) \to \mathsf{CE}(\mathfrak{h})$ be morphisms between the Chevalley–Eilenberg algebras of $\mathfrak{g}$ and $\mathfrak{h}$ and, furthermore, let t^α, b^a and t'^α, b'^a denote the generators of $\mathfrak{g}$ and $\mathfrak{h}$ in degree 1 and 2, respectively. Let the differential be generic as in (3.7). A generic degree -1 map η is given by

$$\eta(t^\alpha) = 0 \ , \qquad \eta(b^a) = \eta^a_\alpha t'^\alpha \ . \tag{2.56}$$

Then, the requirement that η vanishes along $\mathfrak{g}^*[2] \subset \mathsf{W}(\mathfrak{g})$ together with the formula (2.54) subsequently defines η on Qt^α and Qb^a, which we use to calculate

$$\begin{aligned} [Q,\eta]t^\alpha &= -f^\alpha_a \eta^a_\beta\, t'^\beta \ , \\ [Q,\eta]b^a &= -\tfrac{1}{2}\eta^a_\alpha f'^\alpha_{\beta\gamma}\, t'^\beta \wedge t'^\gamma - \eta^a_\alpha f'^\alpha_b\, b'^b + \tfrac{1}{2} f^a_{\alpha b}\eta^b_\gamma (\Psi^\alpha_\beta + \Phi^\alpha_\beta)\, t'^\beta \wedge t'^\gamma \ , \end{aligned} \tag{2.57}$$

where f are the generic structure constants of the 2-term L_∞-algebra $\mathfrak{g}$. The identity $\Phi - \Psi = [Q,\eta]$ therefore is equivalent to

$$\begin{aligned} \Phi^\alpha_\beta - \Psi^\alpha_\beta &= -f^\alpha_a \eta^a_\beta \ , \\ \Phi^a_b - \Psi^a_b &= -\eta^a_\alpha f'^\alpha_b \ , \\ \Phi^a_{[\beta\gamma]} - \Psi^a_{[\beta\gamma]} &= -\eta^a_\alpha f'^\alpha_{[\beta\gamma]} + f^a_{\alpha b}(\Psi + \Phi)^\alpha_{[\beta}\eta^b_{\gamma]} \ . \end{aligned} \tag{2.58}$$

When written in terms of higher brackets and simplifying the last equation by using the first, this just reproduces the more familiar condition for 2-morphisms as given in [6], also cf. [14, Appendix A].

2.5 Lie 2-groups

Just as ordinary Lie algebras are associated with an integrating Lie group, so are Lie 2-algebras connected to the notion of a Lie 2-group. In this thesis we will be mainly concerned with the algebra side, i.e. with n-term L_∞-algebras, as these give the framework in which connections take values in higher gauge theory. Nonetheless, we will recall the basic definitions of Lie 2-groups in this section. The main reference for a comprehensive and detailed discussion is [73].

Definition 2.20 (Weak 2-group)

*A **weak 2-group** $\mathcal{G}$ is a weak monoidal category where all morphisms are invertible and all objects are weakly invertible. That is, a category $\mathcal{G}$ together with*

- *a functor $\otimes : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$, where we write $g \cdot h := \otimes(g,h)$ for both objects and morphisms in $\mathcal{G}$,*
- *an identity object $\mathbb{1} \in \mathcal{G}$,*
- *natural isomorphisms*

$$\begin{aligned} \alpha_{g,h,k} &: (g \cdot h) \cdot k \to g \cdot (h \cdot k) \ , \\ l_g &: \mathbb{1} \cdot g \to g \ , \\ r_g &: g \cdot \mathbb{1} \to g \ , \end{aligned} \tag{2.59}$$

  that satisfy the appropriate higher coherence conditions ,
- *an inverse for every morphism $f \in \mathcal{G}$,*
- *and a weak inverse for every object $x \in \mathcal{G}$, that is, an object $y \in \mathcal{G}$ such that both $x \cdot y$ and $y \cdot x$ are isomorphic to the identity object $\mathbb{1} \in \mathcal{G}$.*

The natural isomorphisms $\alpha_{g,h,k}, l_g$ and r_g represent the possibility that associativity as well as the left and right unit laws may only hold up to isomorphisms — a feature that commonly appears in categorified objects as we discussed previously. For a **strict 2-group** these do hold, i.e. $\alpha_{g,h,k}, l_g$ and r_g are just the identity, and the objects are strictly invertible, i.e. $x \cdot y = y \cdot x = \mathbb{1}$. In order to connect these notions to Lie 2-algebras we need an additional smooth structure which leads to the notion of a Lie 2-group:

Definition 2.21 (Lie 2-group)

*A **Lie 2-group** is a weak 2-group $\mathcal{G} = \mathcal{G}_0 \leftleftarrows \mathcal{G}_1$ where $\mathcal{G}_0$, $\mathcal{G}_1$ and $\mathcal{G}_1 \, {}_s\!\times_t \, \mathcal{G}_1$*[6] *are smooth manifolds and all structure maps, s, t and i, the natural transformations, $\alpha_{g,h,k}, l_g$ and r_g, and the composition of morphisms are smooth.*

Again, for a **strict Lie 2-group** the objects are strictly invertible and the natural transformations appearing in the definition above are just the identity. Such

[6] We write $\mathcal{G}_1 \, {}_s\!\times_t \, \mathcal{G}_1$ for the space of composable morphisms. That is, those morphisms whose source and target maps match up appropriately.

strict Lie 2-group can be straightforwardly differentiated: both the objects and the morphisms form Lie groups that differentiate to their respective Lie algebras, which form the basis of the resulting Lie 2-algebra. See [3] or [6] for details.

A method of differentiating general Lie 2-groups to Lie 2-algebras was given by Ševera [74], which yields the corresponding Lie 2-algebras in the form of 2-term L_∞-algebras.

Methods of integrating arbitrary n-term L_∞-algebras have also been considered, see e.g. [75]. In the strict case, one can also straightforwardly integrate, reversing the process described above. The integration of [75], however, usually yields a different result, which is only categorically equivalent to that of the straightforward integration.

2.6 Example: the string Lie 2-algebra

The main example of L_∞-algebras relevant throughout this thesis is that of the string Lie 2-algebra[7]. The corresponding Lie 2-group is the string group $\mathsf{String}(n)$ that sits in the sequence

$$\ldots \longrightarrow \mathsf{String}(n) \longrightarrow \mathsf{Spin}(n) \longrightarrow \mathsf{Spin}(n) \longrightarrow \mathsf{SO}(n) \longrightarrow \mathsf{O}(n)\ . \tag{2.60}$$

This sequence is known as the Whitehead tower and is achieved by subsequently removing the lowest homotopy group: $\pi_0(\mathsf{O}(n))$ is removed in the step from $\mathsf{O}(n)$ to $\mathsf{SO}(n)$, $\pi_1(\mathsf{O}(n))$ in the step to $\mathsf{Spin}(n)$ and $\pi_2(\mathsf{O}(n))$ is already trivial. The string group $\mathsf{String}(n)$ is obtained by removing $\pi_3(\mathsf{O}(n))$. That is, $\mathsf{String}(n)$ is a 3-connected cover of $\mathsf{Spin}(n)$. This definition only determines $\mathsf{String}(n)$ up to homotopical equivalence, and, as such, there are a variety of models.

The first such models were given in [76, 77] based on topological groups. We are however interested in the more convenient models of the string group in terms of Lie 2-groups. Many such 2-group models exist, but, in this thesis, those of [78] and [79] will suffice. We will refer to these as the **skeletal model** $\mathsf{String}_{\rm sk}(n)$ and the **loop model** $\mathsf{String}_{\hat\Omega}(n)$, respectively. We will not discuss these 2-groups in detail here, but rather focus on the corresponding 2-term L_∞-algebras as these are

[7] We will often refer to this 2-term L_∞-algebra simply as the string algebra.

of more relevance in the following.

In [80], the skeletal model was differentiated using Ševera's method [74], with the result being the **skeletal model of the string algebra** $\mathfrak{string}_{\rm sk}(n)$. This 2-term L_∞-algebra is given by

$$\mathfrak{string}_{\rm sk}(n) = \big(\ \mathfrak{spin}(n) \xleftarrow{\ 0\ } \mathbb{R}\ \big)\ , \tag{2.61}$$

with non-trivial brackets

$$\begin{aligned} &\mu_2 : \mathfrak{spin}(n) \wedge \mathfrak{spin}(n) \to \mathfrak{spin}(n)\ , && \mu_2(x_1, x_2) = [x_1, x_2]\ ,\\ &\mu_3 : \mathfrak{spin}(n) \wedge \mathfrak{spin}(n) \wedge \mathfrak{spin}(n) \to \mathbb{R}\ , && \mu_3(x_1, x_2, x_3) = (x_1, [x_2, x_3])\ , \end{aligned} \tag{2.62}$$

where $[-,-]$ is the commutator and $(-,-)$ is the Killing form on $\mathfrak{spin}(n)$. As μ_1 vanishes this 2-term L_∞-algebra is indeed skeletal, in the sense discussed in Section 2.1. Clearly, such 2-term L_∞-algebras exist for any metric Lie algebra $\mathfrak{g}$ and we write[8] $\mathfrak{string}_{\rm sk}(\mathfrak{g})$ for the 2-term L_∞-algebra with non-trivial brackets given by (2.62). This algebra can also be seen as a form of central extension of $\mathfrak{g}$ by the canonical co-cycle given by μ_3, cf. [14]. Its Weil algebra is given by generators t^α, b^a of degree 1 and 2, respectively, together with their shifted copies $r^\alpha = \sigma t^\alpha$ and $h^a = \sigma b^a$ of degree 2 and 3. The differential corresponding to (2.62) is then

$$\begin{aligned} Qt^\alpha &= -\tfrac{1}{2} f^\alpha_{\beta\gamma} t^\beta \wedge t^\gamma + r^\alpha\ ,\\ Qb^a &= -\tfrac{1}{6} f^a_{\alpha\beta\gamma} t^\alpha \wedge t^\beta \wedge t^\gamma + h^a\ ,\\ Qr^\alpha &= -f^\alpha_{\beta\gamma} t^\beta \wedge r^\gamma\ ,\\ Qh^a &= -\tfrac{1}{2} f^a_{\alpha\beta\gamma} t^\alpha \wedge t^\beta \wedge r^\gamma\ , \end{aligned} \tag{2.63}$$

where $f^\alpha_{\beta\gamma}$ and $f^a_{\alpha\beta\gamma}$ are the structure constants corresponding to μ_2 and μ_3, respectively. In the co-algebra view, the Chevalley–Eilenberg algebra part of this

[8] We will also often drop the reference to $\mathfrak{g}$ when this is clear from context.

corresponds to a co-derivation given by

$$\begin{aligned} \mathcal{D}^1(t_\beta \vee t_\gamma) &= f^\alpha_{\beta\gamma} t_\alpha \ , \\ \mathcal{D}^1(t_\alpha \vee t_\beta \vee t_\gamma) &= f^a_{\alpha\beta\gamma} b_a \ . \end{aligned} \tag{2.64}$$

The loop model on the other hand is a strict 2-group model and can therefore be straightforwardly differentiated yielding the **loop model of the string algebra** $\mathfrak{string}_{\hat{\Omega}}(\mathfrak{g})$, which is a 2-term L_∞-algebra given by

$$\mathfrak{string}_{\hat{\Omega}}(\mathfrak{g}) = \big(\ P_0\mathfrak{g} \xleftarrow{\mu_1} \Omega\mathfrak{g} \oplus \mathbb{R} \ \big) \ , \tag{2.65}$$

where $P_0\mathfrak{g}$ and $\Omega\mathfrak{g}$ are the spaces of based paths and loops in $\mathfrak{g}$, respectively. The non-trivial brackets are

$$\begin{aligned} &\mu_1 : \Omega\mathfrak{g} \oplus \mathbb{R} \to P_0\mathfrak{g} \ , && \mu_1((\lambda, r)) = \lambda \ , \\ &\mu_2 : P_0\mathfrak{g} \wedge P_0\mathfrak{g} \to P_0\mathfrak{g} \ , && \mu_2(\gamma_1, \gamma_2) = [\gamma_1, \gamma_2] \ , \\ &\mu_2 : P_0\mathfrak{g} \otimes (\Omega\mathfrak{g} \oplus \mathbb{R}) \to \Omega\mathfrak{g} \oplus \mathbb{R} \ , \\ &\qquad \mu_2\big(\gamma, (\lambda, r)\big) = \left([\gamma, \lambda] \ , \ -2 \int_0^1 \mathrm{d}\tau \left(\gamma(\tau), \frac{\mathrm{d}}{\mathrm{d}\tau} \lambda(\tau) \right) \right) \ , \end{aligned} \tag{2.66}$$

where $[-,-]$ is the commutator and $(-,-)$ the Killing form of $\mathfrak{g}$. As this is an infinite-dimensional model we would have to introduce additional generators when dualizing to write down the Weil algebra and therefore restrict ourselves to the co-algebra view: we have generators $t_{\alpha\tau}$ of degree 1 for every $\tau \in [0,1]$ in path space and generators $b_{\alpha\tau}$ and b_a of degree 2 for the loops and central directions of $\Omega\mathfrak{g} \oplus \mathbb{R}$. The co-derivation is then given by

$$\begin{aligned} \mathcal{D}^1(b_{\alpha\tau}) &= t_{\alpha\tau} \ , \\ \mathcal{D}^1(t_{\beta\tau} \wedge t_{\gamma\tau}) &= f^\alpha_{\beta\gamma} t_{\alpha\tau} \ , \\ \mathcal{D}^1(t_{\beta\tau} \wedge b_{\gamma\tau}) &= f^\alpha_{\beta\gamma} b_{\alpha\tau} + f^a_{\beta\gamma} b_a \ , \end{aligned} \tag{2.67}$$

where the structure constants $f^\alpha_{\beta\gamma}$ and $f^a_{\beta\gamma}$ again correspond to the brackets μ_i.

The algebra $\mathfrak{string}_{\hat{\Omega}}(\mathfrak{g})$ is categorically equivalent to the skeletal model $\mathfrak{string}_{\rm sk}(\mathfrak{g})$ — a fact, which was shown in [79]. Let us quickly recap the relevant morphisms: in order to show categorical equivalence we need morphisms

$$\mathfrak{string}_{\rm sk}(\mathfrak{g}) \xrightarrow{\Phi} \mathfrak{string}_{\hat{\Omega}}(\mathfrak{g}) \xrightarrow{\Psi} \mathfrak{string}_{\rm sk}(\mathfrak{g}) \tag{2.68}$$

such that their compositions in either direction are connected to the identity via a 2-morphism. Here, the 2-morphism is given in the multi-bracket viewpoint here and is the analogue of the differential graded algebra version discussed in Section 2.4. In this viewpoint, morphisms Φ for 2-term L_∞-algebras are given by a chain map ϕ_1 acting on one generator and a skew-symmetric map ϕ_2 acting on two, cf. (2.28). The chain maps ϕ_1 and ψ_1 are given in the diagram

$$\begin{tikzcd}
\mathbb{R} \arrow[r, hook, "\phi_1"] \arrow[d, "0"] & \Omega\mathfrak{g} \oplus \mathbb{R} \arrow[r, "\psi_1 = {\rm pr}_\mathbb{R}"] \arrow[d, "\mu_1"] & \mathbb{R} \arrow[d, "0"] \\
\mathfrak{g} \arrow[r, hook, "\phi_1 = \cdot f(\tau)"] & P_0\mathfrak{g} \arrow[r, "\psi_1 = \partial"] & \mathfrak{g}\ ,
\end{tikzcd} \tag{2.69}$$

where ${\rm pr}_\mathbb{R}$ is the obvious projection, $\partial : P_0\mathfrak{g} \to \mathfrak{g}$ is the endpoint evaluation and $\cdot f(\tau) : \mathfrak{g} \to P_0\mathfrak{g}$ is the embedding of $x_0 \in \mathfrak{g}$ as the straight line $x(\tau) = x_0 f(\tau)$, where $f : [0,1] \to \mathbb{R}$ is any smooth function[9] with $f(0) = 0$ and $f(1) = 1$. Furthermore, the maps ϕ_2 and ψ_2 read as

$$\begin{aligned}
\phi_2(x_1, x_2) &= \big([x_1, x_2](f(\tau) - f^2(\tau)), 0 \big), \\
\psi_2(x_1, x_2) &= \int_0^1 \mathrm{d}\tau \, (\dot{x}_1, x_2) - (x_1, \dot{x}_2) \ .
\end{aligned} \tag{2.70}$$

It can easily be checked that the corresponding co-algebra morphisms respect the co-derivation by checking the relations in Section 2.2 or Appendix A.2 and, thus, these are indeed L_∞-algebra morphisms. Composing the morphisms using (A.7) one finds that $\Psi \circ \Phi$ is already the identity on $\mathfrak{string}_{\rm sk}(\mathfrak{g})$, whereas $\Phi \circ \Psi$ is not. However,

[9] More precisely, we need f to be lazy in the sense that it is constant in a neighborhood of 0 and 1 . Again, we will suppress this technicality.

the morphism

$$\chi : P_0\mathfrak{g} \to \Omega\mathfrak{g} \oplus \mathbb{R} \ , \quad \chi(\gamma) = \big(\gamma - f(\tau)\partial\gamma, 0\big) \ , \tag{2.71}$$

encodes a 2-morphism $\chi : \Phi \circ \Psi \to \mathrm{id}_{\mathfrak{string}_{\hat{\Omega}}(\mathfrak{g})}$ in the sense of Section 2.4. Indeed, one can easily check that χ satisfies the conditions (2.58). Thus, $\mathfrak{string}_{\mathrm{sk}}(\mathfrak{g})$ and $\mathfrak{string}_{\hat{\Omega}}(\mathfrak{g})$ are equivalent as 2-term L_∞-algebras.

2.7 Cyclic structures for L_∞-algebras

In order to write down Lagrangians from data living in an n-term L_∞-algebra we will need a notion of an inner product: for ordinary Lie algebras we often look at metric matrix Lie algebras with the inner product $(-,-)$ given by the trace. These satisfy a compatibility relation between inner product and Lie bracket,

$$(x_1, [x_2, x_3]) = (x_3, [x_1, x_2]) = (x_2, [x_3, x_1]) \ . \tag{2.72}$$

In the case of L_∞-algebras, we define an analogous concept, which are called cyclic structures.

Definition 2.22 (Cyclic structure for L_∞-algebras)
A ***cyclic structure*** *on an L_∞-algebra $\mathfrak{g}$ is a graded symmetric, non-degenerate bilinear form*

$$\langle -,- \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R} \ , \tag{2.73}$$

satisfying the following compatibility condition for all $x \in \mathfrak{g}$:

$$\begin{aligned} &\langle x_1, \mu_i(x_2, \ldots, x_{i+1})\rangle \\ &= (-1)^{i(|x_1|+1)+|x_{i+1}|(i+|x_1|+\cdots+|x_i|)} \langle x_{i+1}, \mu_i(x_1, \ldots, x_i)\rangle \ . \end{aligned} \tag{2.74}$$

That is, analogously to (2.72), we can cyclically permute the x_i while respecting the usual Koszul convention for permuting graded elements.

These cyclic structures can be seen as naturally arising from an additional symplectic structure on the underlying Q-manifold, cf. Definition 2.12.

Definition 2.23 (Symplectic Q-manifold of degree k)

*A **symplectic Q-manifold of degree** $\mathbf{k}$ is a Q-manifold endowed with a closed, non-degenerate 2-form ω of degree k satisfying $\mathcal{L}_Q\omega = \mathrm{d}\iota_Q\omega = 0$.*

If the degree of ω is odd, such symplectic Q-manifolds are also known as QP-manifolds [81] or P-manifolds [82]. In the general case, they are also called Σ_n-manifolds [83]. Further references are [68] and [64].

A simple example of a symplectic Q-manifold of degree 1 is $T^*[1]M$ with coordinates (x^μ, ξ_μ) of degree 0 and 1, respectively, together with homological vector field $Q = \pi^{\mu\nu}\xi_\mu \frac{\partial}{\partial x^\nu}$ for some anti-symmetric bivector $\pi^{\mu\nu}$. A suitable symplectic form is then given by $\omega = \mathrm{d}x^\mu \wedge \mathrm{d}\xi_\mu$, as, indeed, $\mathcal{L}_Q\omega = \mathrm{d}\iota_Q\omega = \pi^{\mu\nu}\mathrm{d}\xi_\mu \wedge \mathrm{d}\xi_\nu = 0$.

Another example is given by $\mathcal{V}_2 = T^*[2]T[1]M$, as previously discussed at the end of Section 2.2. Choosing the symplectic form

$$\omega = \mathrm{d}x^\mu \wedge \mathrm{d}p_\mu + \mathrm{d}\xi^\mu \wedge \mathrm{d}\xi_\mu \ , \tag{2.75}$$

$\mathcal{V}_2$ becomes a symplectic Q-manifold of degree 2. Indeed, we have

$$\mathcal{L}_{Q_{\mathcal{V}_2}}\omega = \mathrm{d}\iota_{Q_{\mathcal{V}_2}} = \mathrm{d}\xi^\mu \wedge \mathrm{d}p_\mu + \mathrm{d}p_\mu \wedge \mathrm{d}\xi^\mu = 0 \ , \tag{2.76}$$

where $Q_{\mathcal{V}_2}$ is the homological vector field (2.32).

For a generic n-term L_∞-algebra $\mathfrak{g}$ there is not necessarily a symplectic structure on the underlying NQ-manifold and therefore no cyclic structure on $\mathfrak{g}$. This can be amended by minimally extending $\mathfrak{g}$ to the doubled space $T^*[n-1]\mathfrak{g}$: this is concentrated in the same degrees and every generator x of degree k acquires a partner y of degree $n-1-k$. On the corresponding Q-manifold these then have degree $k+1$ and $n-k$ and can be used to form the natural symplectic form of degree $n+1$

$$\omega = \sum_x \mathrm{d}x \wedge \mathrm{d}y \ , \tag{2.77}$$

where the sum runs over the generators of $\mathfrak{g}$. In order to ensure that this is still an L_∞-algebra, or equivalently, that it forms a symplectic Q-manifold of degree $n+1$, we need to find a minimally extended vector field Q_e on the doubled space that squares to zero. This can be constructed as follows. As familiar from classical mechanics, the symplectic form yields an isomorphism between vector fields X_f and corresponding functions f via the equation

$$\iota_{X_f}\omega = \mathrm{d}f \ . \tag{2.78}$$

Furthermore, the symplectic structure induces a Poisson bracket given by

$$\{f, g\} = \iota_{X_f}\iota_{X_g}\omega \ . \tag{2.79}$$

As such we can define a minimal Hamiltonian function $\mathcal{Q}$ that induces an extension of the original homological vector field Q and use the Poisson bracket to find the extended vector field Q_e that automatically squares to zero. In terms of the multi-bracket view of $\mathfrak{g}$ this corresponds to adding the appropriately dualized versions of the brackets μ_i.

Instead of discussing this in detail, let us illustrate this procedure in the example that is relevant for this thesis: an extended version of the skeletal string algebra introduced in the previous section, i.e. Section 2.6. This extended version is based on the space

$$\mathfrak{string}_\mathrm{sk} \xleftarrow{\ \mathrm{id}\ } \mathbb{R} \ = \ \mathfrak{g} \xleftarrow{\ 0\ } \mathbb{R}_r[1] \xleftarrow{\ \mathrm{id}\ } \mathbb{R}_p[2] \ , \tag{2.80}$$

where we add another $\mathbb{R}$ in degree 2 together with $\mu_1 : \mathbb{R}[2] \to \mathbb{R}[1]$ being the identity, while keeping all other maps the same. This extension and reasons for considering it will be introduced in Section 3.5 and will be part of the main relevant example featured in the construction of our 6D Lagrangian in Chapter 5. The corresponding Q-manifold then has local coordinates $x^\alpha \in \mathfrak{g}[1]$, $r \in \mathbb{R}[1]$ and $p \in \mathbb{R}[2]$ of degrees 1, 2 and 3, respectively. In these coordinates, the homological vector field Q is given by

$$Q = -\tfrac{1}{2} f^\alpha_{\beta\gamma} x^\beta x^\gamma \frac{\partial}{\partial x^\alpha} - \tfrac{1}{6} f_{\alpha\beta\gamma} x^\alpha x^\beta x^\gamma \frac{\partial}{\partial r} - p\frac{\partial}{\partial r} \ , \tag{2.81}$$

cf. (2.63). Doubling as sketched above yields

$$
\begin{array}{ccccc}
\mathbb{R}^*_q & \xleftarrow{\mathrm{id}^*} & \mathbb{R}^*_s[1] & & \mathfrak{g}^*[2] \\
\oplus & & \oplus & & \oplus \\
\mathfrak{g} & & \mathbb{R}_r[1] & \xleftarrow{\mathrm{id}} & \mathbb{R}_p[2]\ .
\end{array}
\tag{2.82}
$$

In the corresponding Q-manifold we therefore have additional coordinates $y_\alpha \in \mathfrak{g}^*[3]$, $s \in \mathbb{R}^*[2]$ and $q \in \mathbb{R}^*[1]$ of degrees 3, 2, and 1, respectively. The natural symplectic form of degree 4 reads as

$$
\omega = \mathrm{d}x^\alpha \wedge \mathrm{d}y_\alpha + \mathrm{d}r \wedge \mathrm{d}s + \mathrm{d}p \wedge \mathrm{d}q\ . \tag{2.83}
$$

The induced Poisson bracket is then given by

$$
\begin{aligned}
\{f,g\} = &-f\overleftarrow{\frac{\partial}{\partial y_\alpha}}\overrightarrow{\frac{\partial}{\partial x^\alpha}}g - f\overleftarrow{\frac{\partial}{\partial x^\alpha}}\overrightarrow{\frac{\partial}{\partial y_\alpha}}g - f\overleftarrow{\frac{\partial}{\partial p}}\overrightarrow{\frac{\partial}{\partial q}}g \\
&- f\overleftarrow{\frac{\partial}{\partial q}}\overrightarrow{\frac{\partial}{\partial p}}g + f\overleftarrow{\frac{\partial}{\partial s}}\overrightarrow{\frac{\partial}{\partial r}}g + f\overleftarrow{\frac{\partial}{\partial r}}\overrightarrow{\frac{\partial}{\partial s}}g\ .
\end{aligned}
\tag{2.84}
$$

One can then read off the minimal Hamiltonian function which induces an extension of the original homological vector field (2.81), which is given by

$$
\mathcal{Q}_{\mathrm{min}} = -\tfrac{1}{2}f^\alpha_{\beta\gamma}x^\beta x^\gamma y_\alpha - \tfrac{1}{6}f_{\alpha\beta\gamma}x^\alpha x^\beta x^\gamma s - ps\ . \tag{2.85}
$$

Substituting this into $Q_{\mathrm{e}} = \{\mathcal{Q}_{\mathrm{min}}, -\}$ to find the extended homological vector field yields

$$
\begin{aligned}
Q_{\mathrm{e}} = &-\tfrac{1}{2}f^\alpha_{\beta\gamma}x^\beta x^\gamma\frac{\partial}{\partial x^\alpha} - f^\alpha_{\beta\gamma}x^\beta y_\alpha\frac{\partial}{\partial y_\beta} - \tfrac{1}{6}f_{\alpha\beta\gamma}x^\alpha x^\beta x^\gamma\frac{\partial}{\partial s} \\
&- \tfrac{1}{2}f_{\alpha\beta\gamma}x^\beta x^\gamma s\frac{\partial}{\partial y_\alpha} - p\frac{\partial}{\partial r} - s\frac{\partial}{\partial q}\ .
\end{aligned}
\tag{2.86}
$$

One can readily check that $Q_{\mathrm{e}}^2 = 0$ so that this forms an L_∞-algebra. In terms of the multi-bracket viewpoint of the L_∞-algebra this corresponds to adding the dualized versions of the brackets in (2.62) to arrive at the following set of maps:

$$
\begin{aligned}
&\mu_1 : \mathbb{R}_s[1] \to \mathbb{R}_q,\ \mu_1(s) \coloneqq s\ ,\\
&\mu_1 : \mathbb{R}_p[2] \to \mathbb{R}_r[1],\ \mu_1(p) \coloneqq p\ ,\\
&\mu_2 : \mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{g},\ \mu_2(x_1, x_2) \coloneqq [x_1, x_2]\ ,\\
&\mu_2 : \mathfrak{g} \wedge \mathfrak{g}^*[2] \to \mathfrak{g}^*[2],\ \mu_2(x, y) \coloneqq y([-, x])\ ,\\
&\mu_3 : \mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g} \to \mathbb{R}_r[1],\ \mu_3(x_1, x_2, x_3) \coloneqq (x_1, [x_2, x_3])\ ,\\
&\mu_3 : \mathfrak{g} \wedge \mathfrak{g} \wedge \mathbb{R}_s[1] \to \mathfrak{g}^*[2],\ \mu_3(x_1, x_2, s) \coloneqq (-, [x_1, x_2])s\ .
\end{aligned}
\tag{2.87}
$$

The cyclic inner product on $\mathfrak{string}_{\mathrm{sk}} \leftarrow \mathbb{R}[2]$ corresponding to the symplectic form given in (2.83) reads as

$$
\begin{aligned}
&\langle x_1 + q_1 + r_1 + s_1 + p_1 + y_1, x_2 + q_2 + r_2 + s_2 + p_2 + y_2 \rangle\\
&= y_1(x_2) + y_2(x_1) + p_1 q_2 + q_1 p_2 + r_1 s_2 + s_1 r_2\ ,
\end{aligned}
\tag{2.88}
$$

which we will use in the construction of our Lagrangian in Chapter 5.

Chapter 3

Higher Gauge Theory

In this chapter we introduce higher gauge theory in terms of morphisms of differential graded algebras. This approach is a generalization of ideas by Cartan [11, 12] and Atiyah [13], partially due to [84–86] and, to its full extent, due to [14]. We refine these notions by introducing the $\mathfrak{g}$-connection objects given in [14] and then also discuss the principal G-2-bundles given in [7]. Furthermore, we describe the crucial example relevant to this thesis, that is, the twisted string algebra in the skeletal model following the discussion in [14], see also [60, 61]. We also extend this construction to the loop model of the string algebra, as first done in [48]. Lastly, we briefly discuss the local data of generalized higher gauge theory — an idea presented in [49].

3.1 Higher gauge theory from morphisms of differential graded algebras

Ordinary gauge theory describes connections taking values in a Lie algebra living on a principal bundle. As outlined in the introduction, a higher gauge theory should deal with a categorified version and, thus, describe higher connections taking values in categorified Lie algebras living on a higher principal bundle. In Chapter 2 we introduced L_∞-algebras as a natural categorified version of Lie algebras and their Chevalley–Eilenberg algebra as a description in terms of differential graded algebras. As the de Rham complex is also a differential graded algebra, this language provides a combining viewpoint for both Lie algebras and differential forms — the two key ingredients in gauge theory and, by extension to L_∞-algebras, higher gauge theory.

Therefore, this unifying framework allows us to formulate a local description of higher gauge theory using morphisms, as in Definition 2.14, of differential graded algebras. Locally, a flat connection on an open set $U \subset \mathbb{R}^n$ is given by a morphism from the Chevalley–Eilenberg algebra of the relevant L_∞-algebra $\mathfrak{g}$ to the de Rham complex $\Omega^\bullet(U)$, i.e.

$$\Omega^\bullet(U) \xleftarrow{\;A\;} \mathrm{CE}(\mathfrak{g}) \; . \tag{3.1}$$

Explicitly, let $\mathfrak{g}$ be an ordinary Lie algebra with generators t_α. The differential Q of $\mathrm{CE}(\mathfrak{g})$ is just given by

$$Qt^\alpha = -\tfrac{1}{2} f^\alpha_{\beta\gamma} t^\beta \wedge t^\gamma \; , \tag{3.2}$$

where $f^\alpha_{\beta\gamma}$ are the structure constants of $\mathfrak{g}$. The morphism A acts on generators as $t^\alpha \mapsto A^\alpha_\mu \mathrm{d}x^\mu$ and the condition that it respects the differential directly translates to

$$\mathrm{d}A^\alpha = -\tfrac{1}{2} f^\alpha_{\beta\gamma} A^\beta \wedge A^\gamma \; , \tag{3.3}$$

which corresponds to A being flat, i.e. to its curvature vanishing. This procedure readily generalizes to arbitrary n-term L_∞-algebras.

However, in general, one would like to use this formalism to encode non-flat connections as well. This is accomplished by replacing the Chevalley–Eilenberg algebra with its corresponding Weil algebra, see Definition 2.15. For an ordinary Lie algebra $\mathfrak{g}$ this modifies, as before in (2.45), the above differential to be given by

$$Qt^\alpha = -\tfrac{1}{2} f^\alpha_{\beta\gamma} t^\beta \wedge t^\gamma + r^\alpha \qquad \text{and} \qquad Qr^\alpha = -f^\alpha_{\beta\gamma} t^\beta \wedge r^\gamma \; , \tag{3.4}$$

where again $f^\alpha_{\beta\gamma}$ are the structure constants of $\mathfrak{g}$ and $r^\alpha = \sigma t^\alpha$ is the generator of the shifted space $\mathfrak{g}^*[2]$. A connection in gauge theory is then encoded in a morphism from the Weil algebra to the de Rham complex, i.e.

$$\Omega^\bullet(U) \xleftarrow{\;(A,\mathcal{F})\;} \mathrm{W}(\mathfrak{g}) \; , \tag{3.5}$$

which in coordinates is given by $t^\alpha \mapsto A^\alpha_\mu \mathrm{d}x^\mu$ and $r^\alpha \mapsto \mathcal{F}^\alpha_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$. Recall that a morphism of Weil algebras is entirely and uniquely determined by its action on the unshifted copy $\mathfrak{g}^*[1]$ and, therefore, $\mathcal{F}$ is entirely determined by A. The condition that the differential is respected consequently translates to

$$\begin{aligned} \mathcal{F}^\alpha &= \mathrm{d}A^\alpha + \tfrac{1}{2} f^\alpha_{\beta\gamma} A^\beta \wedge A^\gamma \;, \\ \mathrm{d}\mathcal{F}^\alpha &= -f^\alpha_{\beta\gamma} A^\beta \wedge \mathcal{F}^\gamma \;. \end{aligned} \tag{3.6}$$

Thus, we not only incorporate a non-vanishing curvature $\mathcal{F}$ but also conveniently encode its Bianchi identity. For clarity, let us also discuss the next case in the L_∞-algebra hierarchy, that of a 2-term L_∞-algebra $\mathfrak{g}$. Let its Weil algebra be given by coordinates t^α and b^a of degree 1 and 2, respectively, and their shifted copies r^α and h^a of degree 2 and 3. A generic differential is then given by

$$\begin{aligned} Qt^\alpha &= -\tfrac{1}{2} f^\alpha_{\beta\gamma} t^\beta \wedge t^\gamma - f^\alpha_a b^a + r^\alpha \;, \\ Qb^a &= -f^a_{\alpha b} t^\alpha \wedge b^b - \tfrac{1}{6} f^a_{\alpha\beta\gamma} t^\alpha \wedge t^\beta \wedge t^\gamma + h^a \;, \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} Qr^\alpha &= -f^\alpha_{\beta\gamma} t^\beta \wedge r^\gamma + f^\alpha_a h^a \;, \\ Qh^a &= f^a_{\alpha b} r^\alpha \wedge b^b - f^a_{\alpha b} t^\alpha \wedge h^b + \tfrac{1}{2} f^a_{\alpha\beta\gamma} t^\alpha \wedge t^\beta \wedge r^\gamma \;, \end{aligned} \tag{3.8}$$

where the f denote generalized structure constants. A morphism out of the Weil algebra now splits into the following parts: in addition to $t^\alpha \mapsto A^\alpha_\mu \mathrm{d}x^\mu$ we also have a two-form connection given by $b^a \mapsto B^a_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$. These, in turn, determine the curvatures $r^\alpha \mapsto \mathcal{F}^\alpha_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$ and $h^a \mapsto \mathcal{H}^a_{\mu\nu\kappa} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\kappa$. The compatibility with the differential is then expressed as

$$\begin{aligned} \mathcal{F}^\alpha &= \mathrm{d}A^\alpha + \tfrac{1}{2} f^\alpha_{\beta\gamma} A^\beta \wedge A^\gamma + f^\alpha_a B^a \;, \\ \mathcal{H}^a &= \mathrm{d}B^a + f^a_{\alpha b} A^\alpha \wedge B^b + \tfrac{1}{6} f^a_{\alpha\beta\gamma} A^\alpha \wedge A^\beta \wedge A^\gamma \;, \\ \mathrm{d}\mathcal{F}^\alpha &= -f^\alpha_{\beta\gamma} A^\beta \wedge \mathcal{F}^\gamma + f^\alpha_a \mathcal{H}^a \;, \\ \mathrm{d}\mathcal{H}^a &= f^a_{\alpha b} \mathcal{F}^\alpha \wedge B^b - f^a_{\alpha b} A^\alpha \wedge \mathcal{H}^b + \tfrac{1}{2} f^a_{\alpha\beta\gamma} A^\alpha \wedge A^\beta \wedge \mathcal{F}^\gamma \;, \end{aligned} \tag{3.9}$$

so that, now, we have higher curvatures and their corresponding Bianchi identities.

This again straightforwardly generalizes to morphisms

$$\Omega^\bullet(U) \xleftarrow{(A,\mathcal{F})} \mathrm{W}(\mathfrak{g}) \ , \tag{3.10}$$

where $\mathfrak{g}$ is an arbitrary n-term L_∞-algebra and $(A, \mathcal{F})$ are placeholders for all higher connections and curvatures.

Furthermore, gauge transformations can be encoded in flat homotopies between two such gauge configurations [60], i.e. in morphisms

$$\Omega^\bullet(U \times I) \xleftarrow{(A,\mathcal{F})} \mathrm{W}(\mathfrak{g}) \ , \tag{3.11}$$

where $I = [0, 1]$ denotes the interval and $\mathcal{F}$ vanishes on those additional directions. Denoting the coordinate in the additional direction by ρ, the differential on $\Omega^\bullet(U\times I)$ is given by $Q = \mathrm{d}x^\mu\partial_\mu + \mathrm{d}\rho\frac{\partial}{\partial\rho}$. Then, for an ordinary Lie algebra $\mathfrak{g}$, such a morphism is defined on coordinates as $t^\alpha \mapsto A^\alpha_\mu \mathrm{d}x^\mu + \lambda^\alpha \mathrm{d}\rho$ and $r^\alpha \mapsto \mathcal{F}^\alpha_{\mu\nu}\mathrm{d}x^\nu \wedge \mathrm{d}x^\nu$ and respecting the differentials translates to

$$\begin{aligned}
\mathcal{F}^\alpha &= \mathrm{d}A^\alpha + \tfrac{1}{2} f^\alpha_{\beta\gamma} A^\beta \wedge A^\gamma \ , \\
\mathrm{d}\mathcal{F}^\alpha &= -f^\alpha_{\beta\gamma} A^\beta \wedge \mathcal{F}^\gamma \ , \\
\delta_\lambda A^\alpha &= \mathrm{d}\lambda^\alpha + \tfrac{1}{2} f^\alpha_{\beta\gamma} A^\beta \lambda^\gamma \ , \\
\delta_\lambda \mathcal{F}^\alpha &= f^\alpha_{\beta\gamma} \mathcal{F}^\beta \lambda^\gamma \ ,
\end{aligned} \tag{3.12}$$

where the first two lines are as before and the additional equations are the familiar expressions for an infinitesimal gauge transformation with gauge parameter λ. Again, this procedure readily generalizes to higher, arbitrary n-term L_∞-algebras.

3.2 Equivalent formulation from Maurer–Cartan equations

The above gauge potentials, field strengths and gauge transformations can also be derived in a different, more familiar manner as we explain now, following a similar discussion as that of [35], also see [57, 58]. First, note that the tensor product of the de Rham complex $\Omega^\bullet(X)$ with an L_∞-algebra $\mathfrak{g}$ carries itself a natural L_∞-algebra structure. Explicitly, $\tilde{\mathfrak{g}} \coloneqq \Omega^\bullet(X) \otimes \mathfrak{g}$ carries the higher products

$$\tilde{\mu}_i(f_1 \otimes x_1, \ldots, f_i \otimes x_i) \;=\; \begin{cases} (\mathrm{d}f_1) \otimes x_1 + (-1)^{|f_1|} f_1 \otimes \mu_1(x_1) & \text{for} \quad i = 1 \;, \\ \chi \cdot (f_1 \cdots f_i) \otimes \mu_i(x_1, \ldots, x_i) & \text{for} \quad i > 1 \;, \end{cases} \tag{3.13}$$

where μ_i are the higher products in $\mathfrak{g}$, $f_i \in \Omega^\bullet(X)$ are differential forms on X and $x_i \in \mathfrak{g}$. Furthermore, $|\cdot|$ denotes the degree and $\chi = \pm 1$ is the Koszul sign arising from moving the differential forms of $\Omega^\bullet(X)$ past elements of $\mathfrak{g}$. Note that the total degree of an element $f \otimes x$ in $\Omega^\bullet(X) \otimes \mathfrak{g}$ is $|f| - |x|$ and we truncate $\Omega^\bullet(X) \otimes \mathfrak{g}$ to non-negative degrees.

Recall that an element ϕ of an L_∞-algebra $\tilde{\mathfrak{g}}$ is called a **Maurer–Cartan element**, if it satisfies the **homotopy Maurer–Cartan equation**

$$\sum_{i=1}^{\infty} \frac{(-1)^{i(i+1)/2}}{i!} \tilde{\mu}_i(\phi, \ldots, \phi) \;=\; 0 \;. \tag{3.14}$$

This equations is invariant under infinitesimal gauge symmetries parameterized by an element $\lambda \in \tilde{\mathfrak{g}}$ of degree 0 according to

$$\phi \to \phi + \delta\phi \qquad \text{with} \qquad \delta\phi \;=\; \sum_i \frac{(-1)^{i(i-1)/2}}{(i-1)!} \tilde{\mu}_i(\lambda, \phi, \ldots, \phi) \;, \tag{3.15}$$

cf. [35, 57, 58]. Equation (3.14) states that the higher curvature vanishes and therefore, it can be used to identify the correct notion of curvature. Equation (3.15) then gives the appropriate infinitesimal gauge transformations.

Consider such a Maurer–Cartan element ϕ of degree 1 in $\tilde{\mathfrak{g}}$ for a 2-term L_∞-algebra $\mathfrak{g}$. Using equation (3.14) this can then be identified with the curvatures as given in (3.9). Additionally, a degree 0 element $\lambda \in \tilde{\mathfrak{g}}$ together with (3.15) yields infinitesimal gauge transformations as in (3.12). Altogether, this viewpoint recovers the gauge potential, the curvatures and the infinitesimal gauge transformations of Section 3.1.

3.3 Cartan–Ehresmann connections for L_∞-algebras

In Section 3.1 we discussed how the local data of a connection on an open set U can be encoded in a morphism of differential graded algebras, neglecting the full picture for an ordinary principal G-bundle over a general manifold X. Let us now detail how the conditions for an Ehresmann connection imply further restrictions, that can be encoded in this language. For this purpose, recall the Cartan–Ehresmann conditions for a connection A on a principal G-bundle $P \to X$:

(i) The connection $A \in \Omega^1(P, \mathfrak{g})$ induces the Maurer–Cartan form of G on the fibers, i.e.

$$\iota_{\rho_*(x)} A = x, \;\; \forall x \in \mathfrak{g} \;, \tag{3.16}$$

where ι is the contraction operation and $\mathfrak{g}$ is the Lie algebra of G. Here, $\rho : P \times \mathsf{G} \to P$ is the action of G on P and $\rho_* : \mathfrak{g} \to \Gamma(TP)$ maps to the vector field that at each $p \in P$ is the pushforward of $x \in \mathfrak{g} \sim T_e\mathsf{G}$ to the vector $\rho(p, -)_*(x)$. That is, A maps the vertical vector field generated by each $x \in \mathfrak{g}$ back to x, which on a given fiber G corresponds to the Maurer–Cartan form.

(ii) The connection $A \in \Omega^1(P, \mathfrak{g})$ is G-equivariant, i.e.

$$\mathcal{L}_{\rho_*(x)} A = -[x, A], \;\; \forall x \in \mathfrak{g} \;, \tag{3.17}$$

where ρ_* is as above, $\mathcal{L}_{\rho_*(x)}$ is the Lie derivative along $\rho_*(x)$ and $[-,-]$ is the commutator on $\mathfrak{g}$.

In order to formulate these conditions in terms of Q-manifolds, we need a concept of vertical differential forms corresponding to vertical vector fields.

Definition 3.1 (Vertical de Rham complex)
*Let X be a manifold and let $\pi : Y \twoheadrightarrow X$ be a surjective submersion. The **vertical de Rham complex** $\Omega^\bullet_{vert}(Y)$ is the de Rham complex $\Omega^\bullet(Y)$ of Y modulo the differential ideal Y_h generated by those forms that vanish when restricted to the kernel of $\pi_* : \Gamma(TY) \to \Gamma(TX)$.*

Intuitively, the vertical forms are those that have legs only along the vertical vector fields. Additionally, let q be the canonical quotient map and define the differential

$$\mathrm{d}_{\mathrm{vert}} : \Omega^\bullet_{\mathrm{vert}}(Y) \to \Omega^\bullet_{\mathrm{vert}}(Y),\ q(\omega) \mapsto q(\mathrm{d}\omega)\ , \tag{3.18}$$

which promotes $\Omega^\bullet_{\mathrm{vert}}(Y)$ to a differential graded algebra. To see that this is well defined, note that for a form $\alpha \in Y_h$ we have that $\mathrm{d}\alpha \in Y_h$, which is immediate from the formula

$$\begin{aligned} \mathrm{d}\alpha(y_1, \ldots, y_n) = &\sum_{i=1}^{r} (-1)^{i+1} y_i \alpha(y_1, \ldots, \hat{y}_i, \ldots, y_n) \\ &+ \sum_{i<j} (-1)^{i+j} \alpha([y_i, y_j], y_1, \ldots, \hat{y}_i, \ldots, \hat{y}_j, \ldots, y_n)\ , \end{aligned} \tag{3.19}$$

and the fact that for $y_i, y_j \in \ker(\pi_*)$ we have that $[y_i, y_j] \in \ker(\pi_*)$. Then for $\omega, \omega' \in \Omega^\bullet_{\mathrm{vert}}(Y)$ with $q(\omega) = q(\omega')$, i.e. $\omega - \omega' \in Y_h$, we have

$$\mathrm{d}_{\mathrm{vert}} q(\omega) = q(\mathrm{d}\omega) = q(\mathrm{d}\omega' + \mathrm{d}(\omega - \omega')) = q(\mathrm{d}\omega') = \mathrm{d}_{\mathrm{vert}} q(\omega')\ . \tag{3.20}$$

This shows that $\mathrm{d}_{\mathrm{vert}}$ is well defined and, by construction, q is a morphism of differential graded algebras. Note, that if the surjective submersion $\pi : Y \twoheadrightarrow X$ is a cover by open sets we have $\Omega^\bullet_{\mathrm{vert}}(Y) = 0$. However, for a general surjective submersion this is not the case. In particular, we will consider the case where $Y = P$ is the principal G-bundle P itself, in which case $\Omega^\bullet_{\mathrm{vert}}(Y)$ are the forms with legs only along the fibres G.

Given the surjective submersion $\pi : Y = P \twoheadrightarrow X$ consider the square

$$\begin{tikzcd} \Omega^\bullet_{\mathrm{vert}}(Y) & \mathsf{CE}(\mathfrak{g}) \arrow[l, "A_{\mathrm{vert}}"'] \\ \Omega^\bullet(Y) \arrow[u, two heads, "q"] & \mathsf{W}(\mathfrak{g}) \arrow[l, "(A,\mathcal{F})"'] \arrow[u, two heads, "i^*"'] \end{tikzcd}\ . \tag{3.21}$$

Note that the commutativity of this square implies that the composite morphism $\mathrm{W}(\mathfrak{g}) \to \Omega^\bullet(Y) \to \Omega^\bullet_{\mathrm{vert}}(Y)$, i.e. the potential A along the fibres of P, factors along $\mathrm{CE}(\mathfrak{g})$ to give a map A_{vert}. Being a morphism out of $\mathrm{CE}(\mathfrak{g})$, this therefore means that A_{vert} has vanishing curvature corresponding to the vanishing of the curvature of the Maurer–Cartan form. As such, the first Ehresmann condition (3.16) implies the commutativity of (3.21), which is conveniently expressed in the language of differential graded algebras.

Turning to the second Cartan–Ehresmann condition, let us rewrite (3.17) using Cartan's magic formula, which leads to

$$\begin{aligned} 0 &= \mathcal{L}_{\rho_*(x)}A + [x, A] \\ &= \iota_{\rho_*(x)}\mathrm{d}A + \mathrm{d}\iota_{\rho_*(x)}A + [\iota_{\rho_*(x)}A, A] \\ &= \iota_{\rho_*(x)}\mathcal{F} - \tfrac{1}{2}\iota_{\rho_*(x)}[A, A] + \mathrm{d}x + \tfrac{1}{2}\iota_{\rho_*(x)}[A, A] \\ &= \iota_{\rho_*(x)}\mathcal{F}\ , \end{aligned} \tag{3.22}$$

where we use the first Ehresmann condition (3.16) and the fact that x is constant. That is, $\mathcal{F}$ vanishes along the vertical vector fields $\rho_*(x)$ generated by $x \in \mathfrak{g}$. To encode this condition using Q-manifolds, one would like to use $\wedge^\bullet\mathfrak{g}^*[2] \subset \mathrm{W}(\mathfrak{g})$. However, although $\wedge^\bullet\mathfrak{g}^*[2]$ forms a graded subalgebra of $\mathrm{W}(\mathfrak{g})$, it is not a differential graded subalgebra as Q_{W} does not close on it. The algebra of invariant polynomials $\mathrm{inv}(\mathfrak{g})$ in contrast does form such a differential graded subalgebra. Furthermore, the condition (3.22) implies that for invariant polynomials the Lie derivative $\mathcal{L}_{\rho_*(x)} = \mathrm{d}\iota_{\rho_*(x)} + \iota_{\rho_*(x)}\mathrm{d}$ vanishes, as d closes on $\mathrm{inv}(\mathfrak{g})$. Therefore, the invariant polynomials descend down to the base manifold X which can be encoded in the commutativity of the square

$$\begin{tikzcd} \Omega^\bullet(Y) & \mathsf{W}(\mathfrak{g}) \arrow[l, "(A,\mathcal{F})"'] \\ \Omega^\bullet(X) \arrow[u, "\pi^*"] & \mathsf{inv}(\mathfrak{g}) \arrow[l, "<\mathcal{F}>"'] \arrow[u, hook, "p^*"'] \end{tikzcd}\ , \tag{3.23}$$

i.e. the invariant polynomials applied to $\mathcal{F}$ are pullbacks along $\pi : P \to X$, which is a familiar fact from Chern–Weil theory.

Combining (3.21) and (3.23) leads to the following definition, as given in [14, 60]:

Definition 3.2 ($\mathfrak{g}$-connection object)
Let $\mathfrak{g}$ be an n-term L_∞-algebra and let $\pi : Y \twoheadrightarrow X$ be a surjective submersion. A

$\mathfrak{g}$-***connection object*** *is a morphism* $A : W(\mathfrak{g}) \to \Omega^\bullet(Y)$ *for which the diagram*

$$\begin{tikzcd}
\Omega^\bullet_{\mathrm{vert}}(Y) & \mathsf{CE}(\mathfrak{g}) \arrow[l, "A_{\mathrm{vert}}"'] \\
\Omega^\bullet(Y) \arrow[u, "q"] & \mathsf{W}(\mathfrak{g}) \arrow[l, "(A,\mathcal{F})"'] \arrow[u, "i^*"] \\
\Omega^\bullet(X) \arrow[u, "\pi^*"] & \mathsf{inv}(\mathfrak{g}) \arrow[l, "<\mathcal{F}>"'] \arrow[u, hook, "p^*"]
\end{tikzcd} \tag{3.24}$$

commutes.

This is defined for any surjective submersion and arbitrary n-term L_∞-algebra, but for an ordinary Lie algebra and $Y = P$ a principal G-bundle the commutativity is implied by the Cartan–Ehresmann conditions, as above. It is this additional condition that will become crucial in our discussion of twisted string algebras in Section 3.5.

3.4 The global picture: principal G-2-bundles

Having discussed the local data of connections let us now comment on the global picture: what is a suitable higher version of an ordinary principal G-bundle? There are multiple, equivalent approaches to this, see e.g. [8] for a good overview. Here let us focus on principal G-2-bundles as introduced by Wockel [7], as these offer a straightforward categorified analogue to the ordinary case. We collect the relevant definitions here.

Definition 3.3 (Smooth 2-space)
A ***smooth 2-space*** *is a small category* $\mathcal{M} = \mathcal{M}_0 \leftleftarrows \mathcal{M}_1$ *such that* $\mathcal{M}_0$*,* $\mathcal{M}_1$ *and* $\mathcal{M}_1 \,{}_s\!\times_t \mathcal{M}_1$ *are smooth manifolds and all structure maps,* s, t *and* i*, and the composition of morphisms are smooth. A* ***smooth functor*** *between such smooth 2-spaces is a functor* $F : \mathcal{M} \to \mathcal{M}'$ *such that* F_0 *and* F_1 *are smooth maps. Likewise, a* ***smooth natural transformation*** $\alpha : F \to G$ *is a natural transformation which is smooth as a map* $\mathcal{M}_0 \to \mathcal{M}'_1$*.*

Such a smooth 2-space is just a category internal to the category of smooth manifolds and smooth maps and, as such, appear naturally in the definition of Lie

2-groups, cf. Definition 2.21. A trivial example is given by an ordinary manifold M which can be seen as a smooth 2-space $M \leftleftarrows M$ with only identity morphisms and the obvious structure maps.

Definition 3.4 (Smooth G-2-space)
*Let G be a Lie-2-group. A **smooth G-2-space** is a smooth 2-space $\mathcal{M}$ together with a smooth functor $\rho : \mathcal{M} \times \mathsf{G} \to \mathcal{M}$ and smooth natural isomorphisms*

$$\begin{aligned} \nu &: \rho \circ (\rho \times \mathrm{id}_{\mathsf{G}}) \to \rho \circ (\mathrm{id}_{\mathcal{M}} \times \otimes) \ , \\ \xi &: \rho \circ (\mathrm{id}_{\mathcal{M}} \times \mathbb{1}) \to \mathrm{id}_{\mathcal{M}} \ , \end{aligned} \tag{3.25}$$

that satisfy the appropriate higher coherence conditions.

That is, the functor ρ acts like a group action of G on $\mathcal{M}$ and the natural isomorphisms ν and ξ relax the usual compatibility conditions. One can then imitate the case of an ordinary principal bundle and define:

Definition 3.5 (Principal G-2-bundle)
*Let G be a Lie 2-group and $M \leftleftarrows M$ be a smooth 2-space. A **principal G-2-bundle** over M is a locally trivial G-2-space over $M \leftleftarrows M$. More specifically, it is a smooth G-2-space $\mathcal{P}$ with an open cover $(U_i)_{i\in I}$ of M such that the restrictions $\mathcal{P}|_{U_i}$ are equivalent to the product $U_i \times \mathsf{G}$ as smooth G-2-spaces, i.e. the equivalences are equivariant with respect to the G-action.*

These principal G-2-bundles are classified by non-abelian higher Čech cohomology, just as ordinary principal bundles are classified by Čech co-cycles g_{ij} satisfying the co-cycle condition $g_{ij}g_{jk} = g_{ik}$. A way to look at ordinary co-cycles that allows for a natural generalization is the following.

Given a surjective submersion $\pi : Y \twoheadrightarrow X$ we can define the smooth 2-space $\check{\mathscr{C}}(Y) = Y \leftleftarrows Y^{[2]}$, usually called Čech groupoid, where $Y^{[2]} = Y \times_X Y$ is the fiber product and the structure maps are given by the obvious projections. Composition is given by $(y_1, y_2) \circ (y_2, y_3) = (y_1, y_3)$. Thus, morphisms are invertible with $(y_1, y_2)^{-1} = (y_2, y_1)$ and we, in fact, have a groupoid. The transition functions of a principal G-bundle are then the same as smooth functors $g : \check{\mathscr{C}}(Y) \to \mathsf{BG}$, where $\mathsf{BG} = * \leftleftarrows \mathsf{G}$ is the group seen as a smooth one-object category. The co-cycle condition translates to the compatibility of functors with composition.

This picture readily extends to categorified groups, where we consider higher functors between the Čech groupoid, trivially regarded as a higher category, and the delooping BG of a higher Lie group G. In the case of principal G-2-bundles this results in maps g_{ij} and h_{ijk} on double and triple overlaps, respectively, which describe the co-cycle data and satisfy generalized co-cycles conditions. Furthermore, a global notion of connections on principal G-2-bundles is also available, see [87].

In the abelian case, i.e. for $\mathsf{G} = * \leftleftarrows \mathsf{U}(1) =: \mathsf{BU}(1)$, the principal G-2-bundles defined in this way are equivalent to bundle gerbes [88, 89]. We recall the definition here.

Definition 3.6 (Bundle gerbes)
A ***bundle gerbe*** *over M consists of a surjective submersion $\pi : Y \twoheadrightarrow M$ and a $\mathsf{U}(1)$-principal bundle over the fibered product $Y^{[2]}$ together with a bundle gerbe multiplication. That is, an isomorphism*

$$m : P_{(y_1,y_2)} \otimes P_{(y_2,y_3)} \to P_{(y_1,y_3)} \tag{3.26}$$

of the $\mathsf{U}(1)$-fibres $P_{(-,-)}$ over $Y^{[2]}$ that is required to be associative over $Y^{[4]}$.

When considering the surjective submersion $\pi : Y_{\mathcal{U}} \twoheadrightarrow M$ given by an ordinary cover $Y_{\mathcal{U}}$, we have that $Y_{\mathcal{U}}^{[2]}$ and $Y_{\mathcal{U}}^{[3]}$ are just the double and triple overlaps, respectively, and the bundle multiplication m defines a co-cycle $g_{\alpha\beta\gamma} : U_\alpha \cap U_\beta \cap U_\gamma \to \mathsf{U}(1)$ describing the bundle gerbe — a formulation more familiar to physicists. In fact, these bundle gerbes are, in general, described by a characteristic class in $H^3(X,\mathbb{Z})$, called the Dixmier–Douady class, which is the analogue of the first Chern class of line bundles in $H^2(X,\mathbb{Z})$.

3.5 Example: twisted string Lie 2-algebras

Let us now return to the main example relevant in this thesis: the string algebra discussed in Section 2.6 and its two equivalent guises, the skeletal model $\mathfrak{string}_{\mathrm{sk}}$ and the loop model $\mathfrak{string}_{\hat{\Omega}}$. Having discussed the data of a connection for a general L_∞-algebra and, in particular, the definition of a $\mathfrak{g}$-connection object, see Definition 3.2, we can now investigate the consequences on the string algebra that we want to consider. One philosophy in defining higher connections and curvatures is that they should be in a sense a lift of an ordinary connection. It is this philosophy that we want to apply here, following the ideas outlined in [14].

Recall, that the string algebra $\mathfrak{string}(\mathfrak{g})$ sits above the spin algebra $\mathfrak{g}$ in the sequence (2.60), known as the Whitehead tower. As such, one can expect a $\mathfrak{string}(\mathfrak{g})$-connection to be a lift of a corresponding $\mathfrak{g}$-connection. Let us consider the skeletal model $\mathfrak{string}_{\mathrm{sk}}(\mathfrak{g})$, which exhibits the obvious inclusion $\mathsf{CE}(\mathfrak{string}_{\mathrm{sk}}) \hookleftarrow \mathsf{CE}(\mathfrak{g})$. Then, given a $\mathfrak{g}$-connection object

$$\begin{tikzcd}
\Omega^\bullet_{\mathrm{vert}}(Y) & \mathsf{CE}(\mathfrak{g}) \arrow[l, "A_{\mathrm{vert}}"'] \\
\Omega^\bullet(Y) \arrow[u] & \mathsf{W}(\mathfrak{g}) \arrow[l, "(A,\mathcal{F})"'] \arrow[u] \\
\Omega^\bullet(X) \arrow[u] & \mathsf{inv}(\mathfrak{g}) \arrow[l, "<\mathcal{F}>"'] \arrow[u]
\end{tikzcd} \tag{3.27}$$

one can ask whether or not this lifts to a $\mathfrak{string}_{\mathrm{sk}}$-connection object

$$\begin{tikzcd}
\mathsf{CE}(\mathfrak{string}_{\mathrm{sk}}) \arrow[dr, dashed] & & \mathsf{CE}(\mathfrak{g}) \arrow[ll, hook'] \arrow[dl, "A_{\mathrm{vert}}"] \\
& \Omega^\bullet_{\mathrm{vert}}(Y) & \\
\mathsf{W}(\mathfrak{string}_{\mathrm{sk}}) \arrow[uu] \arrow[dr, dashed] & & \mathsf{W}(\mathfrak{g}) \arrow[ll, hook'] \arrow[uu] \arrow[dl, "(A,\mathcal{F})"] \\
& \Omega^\bullet(Y) \arrow[uu] & \\
\mathsf{inv}(\mathfrak{string}_{\mathrm{sk}}) \arrow[uu] \arrow[dr, dashed] & & \mathsf{inv}(\mathfrak{g}) \arrow[ll, hook'] \arrow[uu] \arrow[dl, "<\mathcal{F}>"] \\
& \Omega^\bullet(X) \arrow[uu] &
\end{tikzcd} . \tag{3.28}$$

In general, this is not possible. However we can instead consider the extended algebra $\mathfrak{string}_{\mathrm{sk}} \longleftarrow \mathbb{R}[2]$ that already appeared in Section 2.7, see (2.80). As can be quickly seen from cohomology, this extended algebra is equivalent to $\mathfrak{g}$ and thus comes with an equivalence $\Phi : \mathsf{CE}(\mathfrak{string}_{\mathrm{sk}} \longleftarrow \mathbb{R}) \longrightarrow \mathsf{CE}(\mathfrak{g})$, which we can employ to extend our diagram to

$$
\begin{tikzcd}
 & & & \mathsf{CE}(\mathfrak{string}_{\mathrm{sk}} \leftarrow \mathbb{R}) \arrow[dlll] \arrow[dl, "\sim"'] \\
\mathsf{CE}(\mathfrak{string}_{\mathrm{sk}}) \arrow[dr, dashed] & & \mathsf{CE}(\mathfrak{g}) \arrow[ll, hook'] \arrow[dl, "A_{\mathrm{vert}}"] & \\
 & \Omega^\bullet_{\mathrm{vert}}(Y) & & \mathsf{W}(\mathfrak{string}_{\mathrm{sk}} \leftarrow \mathbb{R}) \arrow[uu, dotted] \arrow[dlll] \arrow[dl, "\sim"'] \\
\mathsf{W}(\mathfrak{string}_{\mathrm{sk}}) \arrow[uu] \arrow[dr, dashed] & & \mathsf{W}(\mathfrak{g}) \arrow[uu] \arrow[ll, hook'] \arrow[dl, "(A{,}\mathcal{F})"] & \\
 & \Omega^\bullet(Y) \arrow[uu] & & \mathsf{inv}(\mathfrak{string}_{\mathrm{sk}} \leftarrow \mathbb{R}) \arrow[uu, dotted] \arrow[dlll] \arrow[dl, "="'] \\
\mathsf{inv}(\mathfrak{string}_{\mathrm{sk}}) \arrow[uu] \arrow[dr, dashed] & & \mathsf{inv}(\mathfrak{g}) \arrow[uu] \arrow[ll, hook'] \arrow[dl, "<\mathcal{F}>"] & \\
 & \Omega^\bullet(X) \arrow[uu] & &
\end{tikzcd}
\quad . \tag{3.29}
$$

That is, while there may not be a lift to a $\mathfrak{string}_{\mathrm{sk}}$-connection object, we do, at least in principle, always get a $(\mathfrak{string}_{\mathrm{sk}} \leftarrow \mathbb{R})$-connection object. Furthermore, as $\mathsf{CE}(\mathfrak{string}_{\mathrm{sk}} \leftarrow \mathbb{R})$ projects down to $\mathsf{CE}(\mathfrak{string}_{\mathrm{sk}})$, this also conveniently measures the failure to lift this to a $\mathfrak{string}_{\mathrm{sk}}$-connection object. That is, the obstruction is measured by the non-triviality of the component of $A \circ \Phi$ on the extra generator in the additional $\mathbb{R}$, which needs to vanish in order for the lift to exist.

In order to discuss the involved concepts and morphisms in more detail, let the coordinates of $\mathsf{W}(\mathfrak{string}_{\mathrm{sk}} \leftarrow \mathbb{R})$ be given in $\mathfrak{g}^*[1]$ by t^α, b^a and c^μ of degree 1, 2 and 3, respectively, and let their shifted copies in $\mathfrak{g}^*[2]$ be given by r^α, h^a and g^μ of degrees 2, 3 and 4. The differential of $\mathsf{W}(\mathfrak{string}_{\mathrm{sk}} \leftarrow \mathbb{R})$ can then be written as

$$
\begin{aligned}
Qt^\alpha &= -\tfrac{1}{2} f^\alpha_{\beta\gamma} t^\beta \wedge t^\gamma + r^\alpha \ , & Qr^\alpha &= -f^\alpha_{\beta\gamma} t^\alpha \wedge r^\beta \ , \\
Qb^a &= -\tfrac{1}{6} f^a_{\alpha\beta\gamma} t^\alpha \wedge t^\beta \wedge t^\gamma - f^a_\mu c^\mu + h^a \ , & Qh^a &= \tfrac{1}{2} f^a_{\alpha\beta\gamma} t^\alpha \wedge t^\beta \wedge r^\gamma + f^a_\mu g^\mu \ , \\
Qc^\mu &= g^\mu \ , & Qg^\mu &= 0 \ ,
\end{aligned} \tag{3.30}
$$

where f^a_μ is the identity, $f^\alpha_{\beta\gamma}$ encodes the commutator $[-,-]$ and $f^a_{\alpha\beta\gamma}$ encodes $(-,[-,-])$, with $(-,-)$ being the Killing form, also cf. equations (2.62) and (2.81).

Consider first the invariant polynomials of $\mathfrak{string}_{\rm sk}$ itself — they agree with the ones of $\mathfrak{g}$ with one exception. We have that $\mu^a = -\frac{1}{6} f^a_{\alpha\beta\gamma} t^\alpha \wedge t^\beta \wedge t^\gamma$ is a cocycle, as, indeed,

$$Q_{\sf CE}\mu^a = -\tfrac{1}{4} f^a_{\alpha\beta\gamma} f^\gamma_{\delta\epsilon} t^\alpha \wedge t^\beta \wedge t^\delta \wedge t^\epsilon = 0 \ , \tag{3.31}$$

cf. Definition 2.18. In defining $\mathfrak{string}_{\rm sk}$ we introduced the additional generator b^a that explicitly turns μ^a into a coboundary, i.e. $Qb^a = \mu^a$. Therefore, the invariant polynomial P^a that μ^a transgresses to now suspends to and, thus, is horizontally equivalent to 0. A transgression element for P^a and μ^a is given by

$$\mathrm{cs^a} = -\tfrac{1}{6} f^a_{\alpha\beta\gamma} t^\alpha \wedge t^\beta \wedge t^\gamma + \kappa^a_{\alpha\beta} t^\alpha \wedge r^\beta \ , \tag{3.32}$$

where $\kappa^a_{\alpha\beta}$ encodes the Killing form, so that $f^a_{\alpha\beta\gamma} = \kappa^a_{\alpha\delta} f^\delta_{\beta\gamma}$. This leads to the invariant polynomial

$$\begin{aligned}
P^a &= Q(-\tfrac{1}{6} f^a_{\alpha\beta\gamma} t^\alpha \wedge t^\beta \wedge t^\gamma + \kappa^a_{\alpha\beta} t^\alpha \wedge r^\beta) \\
&= \tfrac{1}{4}\kappa^a_{\alpha\epsilon} f^\epsilon_{\beta\zeta} f^\zeta_{\gamma\delta} t^\alpha \wedge t^\beta \wedge t^\gamma \wedge t^\delta - \tfrac{1}{2}\kappa^a_{\delta\gamma} f^\delta_{\alpha\beta} t^\alpha \wedge t^\beta \wedge r^\gamma \\
&\quad + \kappa^a_{\alpha\beta}(Qt^\alpha) \wedge r^\beta + \kappa^a_{\delta\gamma} f^\delta_{\alpha\beta} t^\alpha \wedge t^\beta \wedge r^\gamma \\
&= \kappa^a_{\alpha\beta}(Qt^\alpha) \wedge r^\beta + \tfrac{1}{2}\kappa^a_{\delta\gamma} f^\delta_{\alpha\beta} t^\alpha \wedge t^\beta \wedge r^\gamma \\
&= \kappa^a_{\alpha\beta} r^\alpha \wedge r^\beta \ .
\end{aligned} \tag{3.33}$$

Thus $\mathsf{inv}(\mathfrak{string}_{\rm sk})$ consists of the invariant polynomials in $\mathsf{inv}(\mathfrak{g})$ barring $P^a = \kappa^a_{\alpha\beta} r^\alpha \wedge r^\beta$. Note that this invariant polynomial corresponds to the first Pontryagin class $(\mathcal{F}, \mathcal{F})$.

For $\mathsf{inv}(\mathfrak{string} \leftarrow \mathbb{R})$ this situation changes as we introduce an additional generator c^μ which comes with the additional invariant polynomial g^μ, as $Qg^\mu = 0$. We then have

$$\begin{aligned} P^a &= \kappa^a_{\alpha\beta} r^\alpha \wedge r^\beta \\ &= Q(-\tfrac{1}{6} f^a_{\alpha\beta\gamma} t^\alpha \wedge t^\beta \wedge t^\gamma + \kappa^a_{\alpha\beta} t^\alpha \wedge r^\beta) \\ &= Q(-h^a + f^a_\mu c^\mu + \kappa^a_{\alpha\beta} t^\alpha \wedge r^\beta) \\ &= Q(-h^a + \kappa^a_{\alpha\beta} t^\alpha \wedge r^\beta) + f^a_\mu g^\mu \ , \end{aligned} \tag{3.34}$$

so that, as $-h^a + \kappa^a_{\alpha\beta} t^\alpha \wedge r^\beta$ is in $\ker(i^*)$, now P is horizontally equivalent to the new invariant polynomial g^μ. As such, this restores the missing invariant polynomial and $\mathsf{inv}(\mathfrak{string}_{\rm sk} \leftarrow \mathbb{R})$ is in complete agreement with $\mathsf{inv}(\mathfrak{g})$.

Turning our attention to the Chevalley–Eilenberg algebras we want to write down the equivalences between $\mathsf{CE}(\mathfrak{string}_{\rm sk} \leftarrow \mathbb{R})$ and $\mathsf{CE}(\mathfrak{g})$ explicitly. Let the coordinate in $\mathsf{CE}(\mathfrak{g})$ be t'^α and let the differential Q' be as in (2.45). We then have the morphisms

$$\begin{aligned} &\Phi : \mathsf{CE}(\mathfrak{string}_{\rm sk} \leftarrow \mathbb{R}) \to \mathsf{CE}(\mathfrak{g}) \ , && \Psi : \mathsf{CE}(\mathfrak{g}) \to \mathsf{CE}(\mathfrak{string}_{\rm sk} \leftarrow \mathbb{R}) \ , \\ &\quad t^\alpha \mapsto t'^\alpha \ , && \quad t'^\alpha \mapsto t^\alpha \ , \\ &\quad b^a \mapsto 0 \ , && \\ &\quad c^\mu \mapsto -\tfrac{1}{6} f^\mu_a f^a_{\alpha\beta\gamma} t'^\alpha \wedge t'^\beta \wedge t'^\gamma \ , && \end{aligned} \tag{3.35}$$

where f^μ_a is the inverse of f^a_μ. We check the commutativity of the non-trivial squares,

$$\begin{tikzcd} b^a \arrow[r, "Q"] \arrow[d, "\Phi"] & \mu^a - f^a_\mu c^\mu \arrow[d, "\Phi"] \\ 0 \arrow[r, "Q'"] & \mu^a - \mu^a \ , \end{tikzcd} \qquad \begin{tikzcd} c^\mu \arrow[r, "Q"] \arrow[d, "\Phi"] & 0 \arrow[d, "\Phi"] \\ f^\mu_a \mu^a \arrow[r, "Q'"] & 0 \ , \end{tikzcd} \tag{3.36}$$

so that Φ and Ψ are indeed morphisms of Chevalley–Eilenberg algebras. We have that $\Phi \circ \Psi$ is the identity already and for $\Psi \circ \Phi$ we define the 2-morphism

$$\begin{gathered} \eta : \mathsf{CE}(\mathfrak{string}_{\rm sk} \leftarrow \mathbb{R}) \longrightarrow \mathsf{CE}(\mathfrak{string}_{\rm sk} \leftarrow \mathbb{R}) \ , \\ t^\alpha \mapsto 0 \ , \quad b^a \mapsto 0 \ , \quad c^\mu \mapsto -f^\mu_a b^a \ . \end{gathered} \tag{3.37}$$

One can check, using the formulae in Appendix A.3, that η gives a 2-morphism, as defined in Section 2.4, connecting $\Psi \circ \Phi$ to the identity. Thus, $\mathfrak{g}$ and $\mathfrak{string}_{\rm sk} \leftarrow \mathbb{R}$ are indeed equivalent as L_∞-algebras.

This equivalence also lifts to the level of Weil algebras. With the additional coordinate r'^α in $\mathsf{W}(\mathfrak{g})$ the morphisms are lifted to

$$
\begin{aligned}
&\Phi : \mathsf{W}(\mathfrak{string}_{\rm sk} \leftarrow \mathbb{R}) \to \mathsf{W}(\mathfrak{g})~, && \Psi : \mathsf{W}(\mathfrak{g}) \to \mathsf{W}(\mathfrak{string}_{\rm sk} \leftarrow \mathbb{R})~,\\
&\quad t^\alpha \mapsto t'^\alpha~,\quad r^\alpha \mapsto r'^\alpha~, && \quad t'^\alpha \mapsto t^\alpha~,\quad r'^\alpha \mapsto r^\alpha~,\\
&\quad b^a \mapsto 0~,\quad h^a \mapsto 0~, &&\\
&\quad c^\mu \mapsto -\tfrac{1}{6} f^\mu_a f^a_{\alpha\beta\gamma} t'^\alpha \wedge t'^\beta \wedge t'^\gamma~, &&\\
&\quad g^\mu \mapsto -\tfrac{1}{2} f^\mu_a f^a_{\alpha\beta\gamma} t'^\alpha \wedge t'^\beta \wedge r'^\gamma~. &&
\end{aligned}
\tag{3.38}
$$

We again check the commutativity of the non-trivial squares,

$$
\begin{tikzcd}
b^a \arrow[r, "Q"] \arrow[d, "\Phi"] & \mu^a - f^a_\mu c^\mu + h^a \arrow[d, "\Phi"] & h^a \arrow[r, "Q"] \arrow[d, "\Phi"] & \tfrac{1}{2} f^a_{\alpha\beta\gamma} t^\alpha \wedge t^\beta \wedge r^\gamma + f^a_\mu g^\mu \arrow[d, "\Phi"] \\
0 \arrow[r, "Q'"] & \mu^a - \mu^a~, & 0 \arrow[r, "Q'"] & 0~, \\
c^\mu \arrow[r, "Q"] \arrow[d, "\Phi"] & g^\mu \arrow[d, "\Phi"] & g^\mu \arrow[r, "Q"] \arrow[d, "\Phi"] & 0 \arrow[d, "\Phi"] \\
f^\mu_a \mu^a \arrow[r, "Q'"] & \Phi(g^\mu)~, & \Phi(g^\mu) \arrow[r, "Q'"] & 0~.
\end{tikzcd}
\tag{3.39}
$$

That is, Φ and Ψ still respect the differential and, hence, are promoted to morphisms of Weil algebras. Again, $\Phi \circ \Psi$ is the identity already and $\Psi \circ \Phi$ can be connected to the identity via the 2-morphism

$$
\begin{gathered}
\eta : \mathsf{W}(\mathfrak{string}_{\rm sk} \leftarrow \mathbb{R}) \longrightarrow \mathsf{W}(\mathfrak{string}_{\rm sk} \leftarrow \mathbb{R})~,\\
t^\alpha \mapsto 0~,\quad b^a \mapsto 0~,\quad c^\mu \mapsto -f^\mu_a b^a~.\\
r^\alpha \mapsto 0~,\quad h^a \mapsto 0~,\quad g^\mu \mapsto f^\mu_a h^a~,
\end{gathered}
\tag{3.40}
$$

and we have that $\mathsf{W}(\mathfrak{string}_{\rm sk} \leftarrow \mathbb{R})$ is equivalent to $\mathsf{W}(\mathfrak{g})$.

Crucially, however, this equivalence does not allow for commutativity in the diagram

$$
\begin{tikzcd}
\mathsf{CE}(\mathfrak{g}) & \mathsf{CE}(\mathfrak{string}_{\mathrm{sk}} \leftarrow \mathbb{R}) \arrow[l] \\
\mathsf{W}(\mathfrak{g}) \arrow[u] & \mathsf{W}(\mathfrak{string}_{\mathrm{sk}} \leftarrow \mathbb{R}) \arrow[l] \arrow[u, dotted] \\
\mathsf{inv}(\mathfrak{g}) \arrow[u] & \mathsf{inv}(\mathfrak{string}_{\mathrm{sk}} \leftarrow \mathbb{R}) \arrow[l] \arrow[u, dotted]
\end{tikzcd}\ ,
\tag{3.41}
$$

which is a necessary part of our lift diagram in (3.29). This can be seen as follows: the invariant polynomial g^μ in $\mathsf{inv}(\mathfrak{string}_{\mathrm{sk}} \leftarrow \mathbb{R})$ is identified with $\kappa_{\alpha\beta} r'^\alpha \wedge r'^\beta$ in $\mathsf{inv}(\mathfrak{g})$ and then mapped to the corresponding element in $\mathsf{W}(\mathfrak{g})$. On the other hand $g^\mu \in \mathsf{W}(\mathfrak{string}_{\mathrm{sk}} \leftarrow \mathbb{R})$ is mapped to $f_{\alpha\beta\gamma} t'^\alpha \wedge t'^\beta \wedge r'^\gamma$ in $\mathsf{W}(\mathfrak{g})$ via Φ, which clearly is not an invariant polynomial.

To alleviate this problem we can modify the Weil algebra differential to be given by

$$
\begin{aligned}
Qt^\alpha &= -\tfrac{1}{2} f^\alpha_{\beta\gamma} t^\beta \wedge t^\gamma + r^\alpha\ , \quad Qr^\alpha = -f^\alpha_{\beta\gamma} t^\alpha \wedge r^\beta\ , \\
Qb^a &= -\tfrac{1}{6} f^a_{\alpha\beta\gamma} t^\alpha \wedge t^\beta \wedge t^\gamma + \kappa^a_{\alpha\beta} t^\alpha \wedge r^\beta - f^a_\mu c^\mu + h^a\ , \\
Qh^a &= -\kappa^a_{\alpha\beta} r^\alpha \wedge r^\beta + f^a_\mu g^\mu\ , \quad Qc^\mu = g^\mu\ , \quad Qg^\mu = 0\ ,
\end{aligned}
\tag{3.42}
$$

so that Qb^a is modified to contain the Chern–Simons element cs^a and, in turn, Qh^a is given by the invariant polynomial P^a. We will call this modified algebra the **twisted string algebra in the skeletal model** and denote it by $\widehat{\mathfrak{string}}_{\mathrm{sk}}(\mathfrak{g})$. Its Chevalley–Eilenberg algebra remains unaffected as the modification $\kappa^a_{\alpha\beta} t^\alpha \wedge r^\beta$ lives in the kernel $\ker(i^*)$ and we have $\mathsf{CE}(\widehat{\mathfrak{string}}_{\mathrm{sk}}) = \mathsf{CE}(\mathfrak{string}_{\mathrm{sk}} \leftarrow \mathbb{R})$. Furthermore, the invariant polynomials in $\mathsf{inv}(\widehat{\mathfrak{string}}_{\mathrm{sk}})$ remain unaffected and, again, agree with those in $\mathsf{inv}(\mathfrak{g})$.

Additionally, the Weil algebra $\mathsf{W}(\widehat{\mathfrak{string}}_{\mathrm{sk}})$ is still equivalent to both $\mathsf{W}(\mathfrak{g})$ and

$\mathsf{W}(\mathfrak{string}_{\rm sk} \leftarrow \mathbb{R})$. Explicitly, the equivalences Φ and Ψ are modified to be

$$\begin{aligned} &\Phi : \mathsf{W}(\mathfrak{string}_{\rm sk} \leftarrow \mathbb{R}) \to \mathsf{W}(\mathfrak{g}) \ , && \Psi : \mathsf{W}(\mathfrak{g}) \to \mathsf{W}(\mathfrak{string}_{\rm sk} \leftarrow \mathbb{R}) \ , \\ &\quad t^\alpha \mapsto t'^\alpha \ , \quad r^\alpha \mapsto r'^\alpha \ , && \quad t'^\alpha \mapsto t^\alpha \ , \quad r'^\alpha \mapsto r^\alpha \ , \\ &\quad b^a \mapsto 0 \ , \quad h^a \mapsto 0 \ , && \\ &\quad c^\mu \mapsto f^\mu_a \mathrm{cs}^a \ , \quad g^\mu \mapsto f^\mu_a P^a \ . && \end{aligned} \tag{3.43}$$

Again, we check the commutativity of the non-trivial squares

$$\begin{tikzcd} b^a \arrow[r, "Q"] \arrow[d, "\Phi"] & \mathrm{cs}^a - f^a_\mu c^\mu + h^a \arrow[d, "\Phi"] \\ 0 \arrow[r, "Q'"] & \mathrm{cs}^a - \mathrm{cs}^a \ , \end{tikzcd} \qquad \begin{tikzcd} h^a \arrow[r, "Q"] \arrow[d, "\Phi"] & -P^a + f^a_\mu g^\mu \arrow[d, "\Phi"] \\ 0 \arrow[r, "Q'"] & -P^a + P^a \ , \end{tikzcd}$$
$$\begin{tikzcd} c^\mu \arrow[r, "Q"] \arrow[d, "\Phi"] & g^\mu \arrow[d, "\Phi"] \\ f^\mu_a \mathrm{cs}^a \arrow[r, "Q'"] & f^\mu_a P^a \ , \end{tikzcd} \qquad \begin{tikzcd} g^\mu \arrow[r, "Q"] \arrow[d, "\Phi"] & 0 \arrow[d, "\Phi"] \\ f^\mu_a P^a \arrow[r, "Q'"] & 0 \ . \end{tikzcd} \tag{3.44}$$

which shows that the differentials are still respected. The 2-morphism η, meanwhile, remains unmodified and connects $\Psi \circ \Phi$ to the identity, while $\Phi \circ \Psi$ still trivially gives the identity.

As g^μ is now mapped to the invariant polynomial P^a, this twisted algebra now allows for the diagram

$$\begin{tikzcd} \mathsf{CE}(\mathfrak{g}) & \mathsf{CE}(\widehat{\mathfrak{string}}_{\rm sk}) \arrow[l] \\ \mathsf{W}(\mathfrak{g}) \arrow[u] & \mathsf{W}(\widehat{\mathfrak{string}}_{\rm sk}) \arrow[l] \arrow[u] \\ \mathsf{inv}(\mathfrak{g}) \arrow[u] & \mathsf{inv}(\widehat{\mathfrak{string}}_{\rm sk}) \arrow[l] \arrow[u] \ , \end{tikzcd} \tag{3.45}$$

to commute and for the lift diagram

$$
\begin{tikzcd}
 & & & \mathsf{CE}(\widehat{\mathfrak{string}}_{\mathrm{sk}}) \arrow[dlll] \arrow[dl, "\sim"'] \\
\mathsf{CE}(\mathfrak{string}_{\mathrm{sk}}) \arrow[dr, dashed] & & \mathsf{CE}(\mathfrak{g}) \arrow[ll, hook'] \arrow[dl, "A_{\mathrm{vert}}"'] & \\
 & \Omega^\bullet_{\mathrm{vert}}(Y) & & \mathsf{W}(\widehat{\mathfrak{string}}_{\mathrm{sk}}) \arrow[uu] \arrow[dlll] \arrow[dl, "\sim"'] \\
\mathsf{W}(\mathfrak{string}_{\mathrm{sk}}) \arrow[uu] \arrow[dr, dashed] & & \mathsf{W}(\mathfrak{g}) \arrow[uu] \arrow[ll, hook'] \arrow[dl, "(A{,}\mathcal{F})"'] & \\
 & \Omega^\bullet(Y) \arrow[uu] & & \mathsf{inv}(\widehat{\mathfrak{string}}_{\mathrm{sk}}) \arrow[uu] \arrow[dlll] \arrow[dl, "="'] \\
\mathsf{inv}(\mathfrak{string}_{\mathrm{sk}}) \arrow[uu] \arrow[dr, dashed] & & \mathsf{inv}(\mathfrak{g}) \arrow[uu] \arrow[ll, hook'] \arrow[dl, "<\mathcal{F}>"'] & \\
 & \Omega^\bullet(X) \arrow[uu] & &
\end{tikzcd}
\quad . \tag{3.46}
$$

to be well-defined. The obstruction for the lift to extend to $\mathfrak{string}_{\mathrm{sk}}$ itself is therefore given by the component of the connection on c^μ and g^μ: the first Pontryagin class $(\mathcal{F}, \mathcal{F})$ is required to vanish globally.

In summary, we replace the string algebra $\mathfrak{string}_{\mathrm{sk}}$ with the twisted string algebra $\widehat{\mathfrak{string}}_{\mathrm{sk}}$ as this allows for a consistent lift of a $\mathfrak{g}$-connection. Even though all of $\mathsf{W}(\mathfrak{g})$, $\mathsf{W}(\mathfrak{string}_{\mathrm{sk}} \leftarrow \mathbb{R})$ and $\mathsf{W}(\widehat{\mathfrak{string}}_{\mathrm{sk}})$ are equivalent as L_∞-algebras and we should, in principle, be free to choose any equivalent description, it is only $\mathsf{W}(\widehat{\mathfrak{string}}_{\mathrm{sk}})$ that provides a suitable lift. This is due to the fact that the equivalences given in (3.43) additionally preserve the commutativity of the whole sequence in (3.45). Indeed, we need to restrict ourselves to such equivalences.

The morphisms in (3.43) mix the components in $\mathfrak{g}^*[1]$ and $\mathfrak{g}^*[2]$ of $\mathsf{W}(\widehat{\mathfrak{string}}_{\mathrm{sk}})$, which leads to modified expressions for the curvatures for the twisted string algebra. Let us summarize the relevant data for $\widehat{\mathfrak{string}}_{\mathrm{sk}}$ in the multi-bracket point of view here. The underlying space is given by

$$
\widehat{\mathfrak{string}}_{\mathrm{sk}}(\mathfrak{g}) = \big(\ \mathfrak{g} \xleftarrow{\ 0\ } \mathbb{R}[1] \xleftarrow{\ \mathrm{id}\ } \mathbb{R}[2]\ \big)\ , \tag{3.47}
$$

on which we have the following maps

$$
\begin{aligned}
\kappa &: \mathfrak{g}\otimes\mathfrak{g}\to\mathbb{R}[1]\ , & \kappa(x_1,x_2) &= (x_1,x_2)\ ,\\
\mu_1 &: \mathbb{R}[2]\to\mathbb{R}[1]\ , & \mu_1(r) &= r\ ,\\
\mu_2 &: \mathfrak{g}\wedge\mathfrak{g}\to\mathfrak{g}\ , & \mu_2(x_1,x_2) &= [x_1,x_2]\ ,\\
\mu_3 &: \mathfrak{g}\wedge\mathfrak{g}\wedge\mathfrak{g}\to\mathbb{R}[1]\ , & \mu_3(x_1,x_2,x_3) &= \kappa(x_1,[x_2,x_3])\ ,
\end{aligned}
\tag{3.48}
$$

where $[-,-]$ is the commutator and $(-,-)$ is the Killing form on $\mathfrak{g}$. The curvatures are given by

$$
\begin{aligned}
\mathcal{F} &= \mathrm{d}A+\tfrac{1}{2}\mu_2(A,A)\ ,\\
\mathcal{H} &= \mathrm{d}B+\tfrac{1}{6}\mu_3(A,A,A)-\kappa(A,\mathcal{F})+\mu_1(C)\ ,\\
\mathcal{G} &= \mathrm{d}C\ ,
\end{aligned}
\tag{3.49}
$$

with their Bianchi identities given by

$$
\begin{aligned}
\mathrm{d}\mathcal{F} &= -\mu_2(A,\mathcal{F})\ ,\\
\mathrm{d}\mathcal{H} &= -\kappa(\mathcal{F},\mathcal{F})+\mu_1(\mathcal{G})\ ,\\
\mathrm{d}\mathcal{G} &= 0\ .
\end{aligned}
\tag{3.50}
$$

Note that the Bianchi identity for $\mathcal{H}$ now is the Green–Schwarz anomaly cancelation condition, see e.g. [60, 90]. The upshot of this modification will become clear later, when the twisted string algebra plays a crucial role in our constructions in Chapters 4 and 5.

There is an analogous story for the loop model of the string algebra which leads to the **twisted string algebra in the loop model**, denoted $\widehat{\mathfrak{string}}_{\hat{\Omega}}(\mathfrak{g})$: we extend by an additional $\mathbb{R}$ to arrive at the underlying space

$$
\widehat{\mathfrak{string}}_{\hat{\Omega}}(\mathfrak{g}) = \left(\, P_0\mathfrak{g} \xleftarrow{\ \mu_1\ } \Omega\mathfrak{g}\oplus\mathbb{R} \xleftarrow{\ \mathrm{id}\ } \mathbb{R}\,\right)\ ,
\tag{3.51}
$$

which again is equivalent to $\mathfrak{g}$, as can be seen from cohomology. The role of the cocycle $\mu_3(A,A,A)$ is now played by the cocycle $\mu_2(A,B)$ and we are again required to introduce an additional term in the curvature 3-form $\mathcal{H}$ to allow for the commutativity of the analogous diagrams. This, again, causes the Bianchi identity of $\mathcal{H}$ to be given by the invariant polynomial this cocycle transgresses to. In total we, in addition to the brackets given in (2.66), arrive at the maps

$$\begin{aligned}
&\mu_1 : \mathbb{R} \to \Omega\mathfrak{g} \oplus \mathbb{R}\ , && \mu_1(r) = (0, r)\ ,\\
&\kappa : P_0\mathfrak{g} \otimes P_0\mathfrak{g} \to \Omega\mathfrak{g} \oplus \mathbb{R}\ , && \kappa(\gamma_1, \gamma_2) = \Big(\chi([\gamma_1, \gamma_2])\,,\, 2\int_0^1 \mathrm{d}\tau(\dot{\gamma}_1, \gamma_2)\Big)\ ,
\end{aligned} \tag{3.52}$$

where χ was defined before in (2.71). Here, κ is now a more general map playing the role of the Killing form[1]. Furthermore, analogously to the identity $\mu_3 = \kappa \circ \mu_2$, we now have $\mu_2 = \kappa \circ \mu_1$. The curvatures are given by

$$\begin{aligned}
\mathcal{F} &= \mathrm{d}A + \tfrac{1}{2}\mu_2(A, A) + \mu_1(B)\ ,\\
\mathcal{H} &= \mathrm{d}B + \mu_2(A, B) - \kappa(A, \mathcal{F}) + \mu_1(C)\ ,\\
\mathcal{G} &= \mathrm{d}C\ ,
\end{aligned} \tag{3.53}$$

together with their Bianchi identities

$$\begin{aligned}
\mathrm{d}\mathcal{F} &= -\mu_2(A, \mathcal{F}) + \mu_1(\kappa(A, \mathcal{F})) + \mu_1(\mathcal{H})\ ,\\
\mathrm{d}\mathcal{H} &= -\kappa(\mathcal{F}, \mathcal{F}) + \mu_1(\mathcal{G})\ ,\\
\mathrm{d}\mathcal{G} &= 0\ .
\end{aligned} \tag{3.54}$$

After being twisted the loop string algebra is now again equivalent to the twisted skeletal version. The equivalence is realized by the same maps as before, that is, by the maps given in (2.69) and (2.70).

[1] In a slight abuse of notation we use κ both in the skeletal and in the loop model, refering to different maps. However, from context it will always be clear which map is being used.

3.6 Generalized higher gauge theory

We have introduced higher gauge theory in the form of morphisms from the Weil algebra $\mathsf{W}(\mathfrak{g})$ to the de Rham complex $\Omega^\bullet(X)$. Here, we allowed for a general n-term L_∞-algebra in order to move from ordinary gauge theory to higher gauge theory. Another possibility of generalizing is to replace $\Omega^\bullet(X)$ by a higher space — an idea which we first introduced in [49] and call **generalized higher gauge theory**.

To this end, recall that $\Omega^\bullet(X)$ can be identified with the functions on $T[1]X$, where the coordinates (x^μ, ξ^μ) of degrees 0 and 1 are identified with the coordinate x^μ and the forms $\mathrm{d}x^\mu$, respectively. This now offers a straightforward generalization and we can replace $T[1]X$ with $\mathcal{V}_2 \coloneqq T^*[2]T[1]M$, which we introduced at the end of Section 2.2. A morphism from $\mathsf{W}(\mathfrak{g})$ of an n-term L_∞-algebra to $\mathcal{V}_2$ then yields generalized higher gauge theory.

Let us again focus on the constructive example where $\mathfrak{g}$ is a 2-term L_∞-algebra $\mathfrak{g}_0 \leftarrow \mathfrak{g}_1$, so that $\mathsf{W}(\mathfrak{g})$ is generated by t^α and b^a of degrees 1 and 2 together with their shifted copies r^α and h^a of degrees 2 and 3. The coordinates on $\mathcal{V}_2$ are given by $(x^\mu, \xi^\mu, \xi_\mu, p_\mu)$ of degrees 0, 1, 1, and 2, respectively, and its differential is given in (2.32). Sticking with this notation we now write $A_\mu \xi^\mu$ for $A_\mu \mathrm{d}x^\mu$, $\mathcal{F}_{\mu\nu}\xi^\mu\xi^\nu$ for $\mathcal{F}_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$ and similarly for higher connections and curvatures. For convenience we group together the coordinates ξ^μ and ξ_μ to give a single $\xi^M = (\xi^\mu, \xi_\mu)$ of degree 1, where M now runs from 1 to $2\mathrm{dim}(X)$. The connection morphism is then given by

$$\begin{aligned} t^\alpha &\mapsto A^\alpha_M \xi^M \ , \\ b^a &\mapsto \tfrac{1}{2} B^a_{MN}\xi^M\xi^N + B^{a\mu}p_\mu \ , \end{aligned} \tag{3.55}$$

which we can combine to give the generalized 2-connection

$$\mathcal{A} = A_\mu \xi^\mu + A^\mu \xi_\mu + \tfrac{1}{2}B_{\mu\nu}\xi^\mu\xi^\nu + B_\mu{}^\nu \xi^\mu \xi_\nu + \tfrac{1}{2}B^{\mu\nu}\xi_\mu\xi_\nu + B^\mu p_\mu \ . \tag{3.56}$$

Here, $A = A_M\xi^M = A_\mu\xi^\mu + A^\mu\xi_\mu$ can now be regarded as the sum of a 1-form and a vector field, which are both $\mathfrak{g}_0[1]$-valued. Similarly, B consists of a 2-form, a bivector, a tensor of rank (1,1) and a vector field, all taking values in $\mathfrak{g}_1[1]$. Similarly, the curvature part of the morphism is given by

$$
\begin{aligned}
r^\alpha &\mapsto \tfrac{1}{2}\mathcal{F}^\alpha_{MN}\xi^M\xi^M + \mathcal{F}^{\alpha\mu}p_\mu \ , \\
h^a &\mapsto \tfrac{1}{3!}H^a_{MNK}\xi^M\xi^N\xi^K + H^{a\mu}_{\ \ M}\xi^M p_\mu \ ,
\end{aligned}
\tag{3.57}
$$

which combines to the generalized 2-curvature

$$
\begin{aligned}
\mathcal{F} = {}& \tfrac{1}{2}\mathcal{F}_{\mu\nu}\xi^\mu\xi^\nu + \mathcal{F}_\mu{}^\nu\xi^\mu\xi_\nu + \tfrac{1}{2}\mathcal{F}^{\mu\nu}\xi_\mu\xi_\nu + \mathcal{F}^\mu p_\mu \\
& + \tfrac{1}{3!}H_{\mu\nu\kappa}\xi^\mu\xi^\nu\xi^\kappa + \tfrac{1}{2}H_{\mu\nu}{}^\kappa\xi^\mu\xi^\nu\xi_\kappa + \tfrac{1}{2}H_\mu{}^{\nu\kappa}\xi^\mu\xi_\nu\xi_\kappa \\
& + \tfrac{1}{3!}H^{\mu\nu\kappa}\xi_\mu\xi_\nu\xi_\kappa + H_\mu{}^\nu\xi^\mu p_\nu + H^{\mu\nu}\xi_\mu p_\nu \ .
\end{aligned}
\tag{3.58}
$$

The components of $\mathcal{F}$ are computed to be

$$
\begin{aligned}
\mathcal{F} = {}& \left(\frac{\partial}{\partial x^\mu}A_\nu + \tfrac{1}{2}\mu_2(A_\mu, A_\nu) + \tfrac{1}{2}\mu_1(B_{\mu\nu})\right)\xi^\mu\xi^\nu + \\
& + \left(\frac{\partial}{\partial x^\mu}A^\nu + \mu_2(A_\mu, A^\nu) + \mu_1(B_\mu{}^\nu)\right)\xi^\mu\xi_\nu + \\
& + \left(\tfrac{1}{2}\mu_2(A^\mu, A^\nu) + \tfrac{1}{2}\mu_1(B^{\mu\nu})\right)\xi_\mu\xi_\nu + \left(A^\mu + \mu_1(B^\mu)\right)p_\mu + \\
& + \left(-\tfrac{1}{3!}\mu_3(A_\mu, A_\nu, A_\kappa) + \tfrac{1}{2}\mu_2(A_\mu, B_{\nu\kappa}) + \tfrac{1}{2}\frac{\partial}{\partial x^\mu}B_{\nu\kappa}\right)\xi^\mu\xi^\nu\xi^\kappa + \\
& + \left(-\tfrac{1}{2}\mu_3(A_\mu, A_\nu, A^\kappa) + \mu_2(A_\mu, B_\nu{}^\kappa) + \tfrac{1}{2}\mu_2(A^\kappa, B_{\mu\nu}) + \frac{\partial}{\partial x^\mu}B_\nu{}^\kappa\right)\xi^\mu\xi^\nu\xi_\kappa + \\
& + \left(-\tfrac{1}{2}\mu_3(A_\mu, A^\nu, A^\kappa) + \tfrac{1}{2}\mu_2(A_\mu, B^{\nu\kappa}) - \mu_2(A^\nu, B_\mu{}^\kappa) + \tfrac{1}{2}\frac{\partial}{\partial x^\mu}B^{\nu\kappa}\right)\xi^\mu\xi_\nu\xi_\kappa + \\
& + \left(-\tfrac{1}{3!}\mu_3(A^\mu, A^\nu, A^\kappa) + \tfrac{1}{2}\mu_2(A^\mu, B^{\nu\kappa})\right)\xi_\mu\xi_\nu\xi_\kappa + \\
& + \left(\mu_2(A_\mu, B^\nu) + B_\mu{}^\nu + \frac{\partial}{\partial x^\mu}B^\nu\right)\xi^\mu p_\nu + \left(\mu_2(A^\mu, B^\nu) + B^{\mu\nu}\right)\xi_\mu p_\nu \ .
\end{aligned}
\tag{3.59}
$$

Again, flat homotopies between two such gauge configurations encode generalized gauge transformations, cf. Section 3.1. That is, we extend to $T^*[2]T[1]M \times T[1]I$, where we introduce additional coordinates (r, ρ) of degrees $(0, 1)$ in the new direction and the vector field $Q_{\mathcal{V}_2}$ is amended to

$$\hat{Q}_{\mathcal{V}_2} = \xi^\mu \frac{\partial}{\partial x^\mu} + p_\mu \frac{\partial}{\partial \xi_\mu} + \rho \frac{\partial}{\partial r} \ . \tag{3.60}$$

The connection morphisms are consequently modified to be given by

$$t^\alpha \mapsto \hat{A}^\alpha_M \xi^M + \hat{A}^\alpha_\rho \rho \ , \tag{3.61}$$

$$\hat{a}^*(v^a) \mapsto \tfrac{1}{2} \hat{B}^a_{MN} \xi^M \xi^N + \hat{B}^a_{M\rho} \xi^M \rho + \hat{B}^{a\mu} p_\mu \ , \tag{3.62}$$

which lead to curvature terms along the homotopy direction

$$\begin{aligned}
\mathcal{F}_\perp = &\left(-\frac{\partial}{\partial r} \hat{A}_\mu + \frac{\partial}{\partial x^\mu} \hat{A}_\rho + \mu_1(\hat{B}_{\mu\rho}) + \mu_2(\hat{A}_\mu, \hat{A}_\rho) \right) \xi^\mu \rho \\
&+ \left(-\frac{\partial}{\partial r} \hat{A}^\mu + \mu_1(\hat{B}^\mu{}_\rho) + \mu_2(\hat{A}^\mu, \hat{A}_\rho) \right) \xi_\mu \rho \\
&+ \left(\tfrac{1}{2} \frac{\partial}{\partial r} \hat{B}_{\mu\nu} + \frac{\partial}{\partial x^\mu} \hat{B}_{\nu\rho} + \mu_2(\hat{A}_\mu, \hat{B}_{\nu\rho}) + \tfrac{1}{2} \mu_2(\hat{A}_\rho, \hat{B}_{\mu\nu}) \right. \\
&\qquad\qquad \left. - \tfrac{1}{2} \mu_3(\hat{A}_\mu, \hat{A}_\nu, \hat{A}_\rho) \right) \xi^\mu \xi^\nu \rho \\
&+ \left(\frac{\partial}{\partial r} \hat{B}_\mu{}^\nu + \frac{\partial}{\partial x^\mu} \hat{B}^\nu{}_\rho + \mu_2(\hat{A}_\mu, \hat{B}^\nu{}_\rho) - \mu_2(\hat{A}^\nu, \hat{B}_{\mu\rho}) \right. \\
&\qquad\qquad \left. + \mu_2(\hat{A}_\rho, \hat{B}_\mu{}^\nu) - \mu_3(\hat{A}_\mu, \hat{A}^\nu, \hat{A}_\rho) \right) \xi^\mu \xi_\nu \rho \\
&+ \left(\tfrac{1}{2} \frac{\partial}{\partial r} \hat{B}^{\mu\nu} + \mu_2(\hat{A}^\mu, \hat{B}^\nu{}_\rho) + \tfrac{1}{2} \mu_2(\hat{A}_\rho, \hat{B}^{\mu\nu}) - \tfrac{1}{2} \mu_3(\hat{A}^\mu, \hat{A}^\nu, \hat{A}_\rho) \right) \xi_\mu \xi_\nu \rho \\
&+ \left(B^\mu{}_\rho - \frac{\partial}{\partial r} \hat{B}^\mu - \mu_2(\hat{A}_\rho, \hat{B}^\mu) \right) p_\mu \rho \ .
\end{aligned} \tag{3.63}$$

The requirement that these terms vanish yields the infinitesimal gauge transformations

$$
\begin{aligned}
&\frac{\partial}{\partial r}\hat{A}_\mu = \frac{\partial}{\partial x^\mu}\hat{A}_\rho + \mu_1(\hat{B}_{\mu\rho}) + \mu_2(\hat{A}_\mu, \hat{A}_\rho),\\
&\frac{\partial}{\partial r}\hat{A}^\mu = \mu_1(\hat{B}^\mu{}_\rho) + \mu_2(\hat{A}^\mu, \hat{A}_\rho)\ ,\\
&\frac{\partial}{\partial r}\hat{B}_{\mu\nu} = -2\frac{\partial}{\partial x^\mu}\hat{B}_{\nu\rho} - 2\mu_2(\hat{A}_\mu, \hat{B}_{\nu\rho}) + \mu_2(\hat{A}_\rho, \hat{B}_{\nu\mu}) + \mu_3(\hat{A}_\mu, \hat{A}_\nu, \hat{A}_\rho)\ ,\\
&\frac{\partial}{\partial r}\hat{B}_\mu{}^\nu = -\frac{\partial}{\partial x^\mu}\hat{B}^\nu{}_\rho - \mu_2(\hat{A}_\mu, \hat{B}^\nu{}_\rho) + \mu_2(\hat{A}^\nu, \hat{B}_{\mu\rho}) - \mu_2(\hat{A}_\rho, \hat{B}_\mu{}^\nu) + \mu_3(\hat{A}_\mu, \hat{A}^\nu, \hat{A}_\rho)\ ,\\
&\frac{\partial}{\partial r}\hat{B}^{\mu\nu} = -2\mu_2(\hat{A}^\mu, \hat{B}^\nu{}_\rho) + \mu_2(\hat{A}_\rho, \hat{B}^{\nu\mu}) + \mu_3(\hat{A}^\mu, \hat{A}^\nu, \hat{A}_\rho)\ ,\\
&\frac{\partial}{\partial r}\hat{B}^\mu = B^\mu{}_\rho - \mu_2(\hat{A}_\rho, \hat{B}^\mu)\ ,
\end{aligned}
\tag{3.64}
$$

which are parameterized by a $\mathfrak{g}_0[1]$-valued function $\hat{A}_\rho$, as well as a 1-form $\hat{B}_{\mu\rho}$ and a vector field $\hat{B}^\mu{}_\rho$, both taking values in $\mathfrak{g}_1[1]$.

Note that generalized higher gauge theory contains higher gauge theory. In particular, if we set the fields A^μ, $B_\mu{}^\nu$, $B^{\mu\nu}$ and B^μ to zero, we obtain the usual 2-connection. Analogously, we can restrict the gauge transformations.

Generalized higher gauge theory can also be expressed by the equivalent description in terms of Maurer–Cartan equations as in Section 3.2. Here, we again replace the de Rham complex $\Omega^\bullet(X)$ with $\mathcal{V}_2$, see [49] for details.

In [49] we give possible applications for this generalized version of higher gauge theory: we discuss action functionals arising from Chern–Simons elements of a given cocycle and invariant polynomial, analogous to the AKSZ construction [81], and reformulate the equations for the (2,0) tensor multiplet in six dimensions given in [29] in terms of generalized higher gauge theory.

Chapter 4

The Non-Abelian Self-Dual String

In this chapter we summarize the results of [48], in which we constructed self-dual string equations and gave explicit non-abelian solutions.

4.1 Context

In [48] we aimed to construct a non-abelian version of self-dual strings using the framework of higher gauge theory. These self-dual strings are a higher version of monopoles. Recall that BPS monopoles are objects in 3 dimensions that satisfy the Bogomolny equation

$$\mathcal{F} = *_3(\nabla\Phi) \ , \tag{4.1}$$

where $\mathcal{F}$ is the 2-form curvature of a 1-form curvature, $*_3$ is the Hodge dual in three dimensions and Φ is the Higgs field. From the string theory point of view such monopoles can be described by D1-branes ending on D3-branes [91]. A lift to M-theory yields self-dual strings, which are given by M2-branes ending on M5-branes [92], cf. Section 4.3 for more details. Such self-dual strings should be BPS states in the six-dimensional $\mathcal{N} = (2,0)$ superconformal field theory, often simply referred to as the (2,0)-theory, cf. Section 1.2. Thus, a classical description of these self-dual strings would help advance our understanding of the (2,0)-theory.

Further motivation stems from the development and study of higher integrable models. The BPS monopole equation is an example of a classical integrable system, just as the self-dual Yang–Mills or instanton equations in four dimensions as well as Hitchin's vortex equations. This means that monopoles have rich underlying geometric structures that allow for a relatively explicit description of the solutions and their moduli space. Among these geometric structures are twistor descriptions as well as the Nahm transform which, in an extreme variant, generates solutions to the Bogomolny monopole equation from solutions to a one-dimensional equation via zero-modes of a Dirac operator. Higher and non-abelian generalizations of integrable systems exist, and their moduli spaces have been described using twistor methods [33–36]. The corresponding higher Nahm transform would certainly be very interesting in its own right and may yield further insights into the dualities of M-theory. Most interestingly for mathematicians is that it would provide us with a natural candidate for a categorified Dirac operator, a very important and still missing ingredient in elliptic cohomology.

Lastly, more motivation comes from higher differential geometry. Abelian gerbes have become an important tool in areas such as twisted K-theory and many interesting examples are known, often of relevance in string theory. The situation is very different for non-abelian gerbes: there is a distinct lack of non-trivial and truly non-abelian gerbe with connection that is relevant to string or M-theory beyond the one presented in [48] and discussions in [93, 94]. Without explicit examples, however, it is difficult to develop a mathematical area to its full potential, and this seems to have been a problem of higher gauge theory in the past.

The abelian self-dual strings are solutions to the straightforward generalization of the Bogomolny equation given by

$$\mathcal{H} \coloneqq \mathrm{d}B = *(\mathrm{d}\Phi) \ , \tag{4.2}$$

where $\mathcal{H}$ is a 3-form curvature coming from a 2-form potential B, Φ denotes the Higgs field and $*$ is now the Hodge dual in four dimensions[1]. Solutions to this are known, see [92]. Our aim is to provide a non-abelian generalization with explicit solutions that satisfy the relevant consistency conditions. In analogy with monopoles the non-abelian solutions should be non-singular and approach the abelian self-dual string at infinity. Furthermore, the equations and solutions should be agnostic with respect to categorical equivalence: equivalence classes of solutions should be mapped

[1]In this chapter $*$ will always denote the Hodge dual in four dimensions unless specified otherwise.

to equivalence classes of solutions under such an equivalence.

In constructing the non-abelian self-dual strings there are two main steps: identifying a gauge structure, which should be an L_∞-algebra in which the connections and curvatures take values, and identifying the equations, which suitably generalize (4.2) to the non-abelian case and include constraints for the two-form curvature $\mathcal{F}$. We will address these issues in the following sections.

4.2 Dirac and 't Hooft–Polyakov monopole

Before coming to the self-dual strings let us review a few facts about monopoles, which will serve as the model from which we can draw analogies. As mentioned in the previous section, monopoles are objects in three dimensions that satisfy the Bogomolny equation

$$\mathcal{F} = *_3 \nabla\Phi \ , \tag{4.3}$$

where $\mathcal{F}$ is the two-form curvature, $\nabla = \mathrm{d} + [A, -]$ denotes the usual covariant derivative and Φ is the Higgs field. Here, all of $A, \mathcal{F}$ and Φ take values in a Lie algebra $\mathfrak{g}$ corresponding to a given gauge group G. This equation can be seen as arising from the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}\int\limits_X \mathrm{tr}\left(\mathcal{F} \wedge *\mathcal{F} + \nabla\Phi \wedge *\nabla\Phi\right) . \tag{4.4}$$

Indeed, we can complete the square yielding

$$\begin{aligned} \mathcal{L} &= \tfrac{1}{2}\int\limits_X \mathrm{tr}\left(\mathcal{F} \wedge *\mathcal{F} + \nabla\Phi \wedge *\nabla\Phi\right) \\ &= \tfrac{1}{2}\int\limits_X \mathrm{tr}\left((\mathcal{F} - *\nabla\Phi) \wedge *(\mathcal{F} - *\nabla\Phi) + 2\mathcal{F} \wedge \nabla\Phi\right) \\ &\geq \int\limits_X \mathrm{tr}\left(\mathcal{F} \wedge \nabla\Phi\right) , \end{aligned} \tag{4.5}$$

and the above bound is saturated precisely when the Bogomolny equation (4.3) is satisfied.

The simplest example of such an object is the Dirac monopole whose gauge group is $\mathsf{U}(1)$. Thus, A, $\mathcal{F}$ and Φ take values in $\mathbb{R}$. Let X be $\mathbb{R}^3$ covered by the two charts U_+ and U_-, which we obtain by removing the negative and positive x_3-axis, respectively. In spherical coordinates (r, θ, ϕ) the Dirac monopole is then given by

$$\begin{aligned} A_+ &= \tfrac{im}{2}(1-\cos\theta)\mathrm{d}\phi\ , \\ A_- &= \tfrac{im}{2}(-1-\cos\theta)\mathrm{d}\phi\ , \\ \mathcal{F} &= \tfrac{im}{2}\sin\theta\mathrm{d}\theta\wedge\mathrm{d}\phi\ , \\ \Phi &= -\tfrac{im}{2r}\ , \end{aligned} \tag{4.6}$$

where $m \in \mathbb{Z}$ is the charge of the monopole. The potential $A_\pm$ is singular along the negative (positive) x_3-axis and is therefore only defined on $U_\pm$. As the coordinate r does not appear in the potential or curvature we can consider $A_\pm$ to be defined over S^2 only, on which they give the connection data of an underlying principal bundle. For $m = 1$, this $\mathsf{U}(1)$-bundle P is the Hopf fibration

$$\begin{tikzcd} \mathsf{U}(1) \arrow[r, hook] & S^3 \arrow[d, "\pi"] \\ & S^2\ , \end{tikzcd} \tag{4.7}$$

with transition function $g_{-+} = \exp(i\phi)$ and first Chern number $m = -\frac{i}{2\pi}\int_{S^2}\mathcal{F} = 1$. This bundle is non-trivial corresponding to the singularity of the potentials A_+ and A_- at the south and north poles, respectively.

A further example is given by the 't Hooft–Polyakov monopole [95–97], which is based on the trivial bundle $\tilde{P}$

$$\begin{tikzcd} \mathsf{SU}(2) \arrow[r, hook] & S^2\times\mathsf{SU}(2) \arrow[d, "\mathrm{pr}_1"] \\ & S^2\ , \end{tikzcd} \tag{4.8}$$

where the gauge group is now $\mathsf{SU}(2)$. Extended from S^2 to all of $\mathbb{R}^3$ the potential and Higgs field are given by

$$A = -\tfrac{i}{2}\frac{1-h(r)}{r^2}\epsilon^i{}_{jk}\sigma_i x^k \mathrm{d}x^j \ ,$$
$$\Phi = -\frac{ix^i f(r)}{2r^2}\sigma_i \ , \tag{4.9}$$

where $f(r)$ and $h(r)$ are smooth functions and σ_i denote the usual Pauli matrices. The Bogomolny equation relates $f(r)$ and $h(r)$, which are also required to satisfy certain boundary conditions. In particular, we have that $h(r) \to 0$ as $r \to \infty$.

One can recover the Dirac monopole from the 't Hooft–Polyakov asymptotically. To see this, consider the singular gauge transformation

$$\gamma : \mathbb{R}^3 \to \mathsf{SU}(2) \ , \quad \gamma(x) = \begin{pmatrix} \cos\frac{\theta}{2} & e^{-i\phi}\sin\frac{\theta}{2} \\ -e^{i\phi}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \ , \tag{4.10}$$

which acts on the potential A to give $A' = \gamma A\gamma^{-1} + \gamma\mathrm{d}\gamma^{-1}$ as usual. Applied to the potential $A = -\frac{i}{2}A^j\sigma_j$ of the 't Hooft–Polyakov monopole this yields

$$A^1 = -h(r)(\cos\phi\sin\theta\mathrm{d}\phi + \sin\phi\mathrm{d}\theta) \ ,$$
$$A^2 = -h(r)(\sin\phi\sin\theta\mathrm{d}\phi - \cos\phi\mathrm{d}\theta) \ , \tag{4.11}$$
$$A^3 = -(1-\cos\theta)\mathrm{d}\phi \ ,$$

so that for $r \to \infty$ with $h(r) \to 0$ we recover the Dirac monopole along the third direction.

This can also be seen from a different point of view, in which we embed the bundle P corresponding to the Dirac monopole into the bundle $\tilde{P}$ corresponding to the 't Hooft–Polyakov monopole via a bundle morphism. Recall that such a bundle morphism involves a map $f_p : P \to \tilde{P}$, which is the identity on the base manifold S^2, together with a group homomorphism $\varphi : \mathsf{U}(1) \to \mathsf{SU}(2)$, such that we have

$$f_p \circ R_g = \tilde{R}_{\varphi(g)} \circ f_p \ , \tag{4.12}$$

where R and $\tilde{R}$ denote the right action on P and $\tilde{P}$, respectively. We pick

$$\varphi(e^{i\alpha}) = e^{\frac{i\alpha}{2}x^j\sigma_j}\ , \tag{4.13}$$

and on the trivialization $U_\pm$ then write

$$f_p(x, e^{i\alpha}) = (x, G_\pm(x, e^{i\alpha}))\ , \tag{4.14}$$

where $G_\pm : P \to \mathsf{SU}(2)$ and is given by

$$\begin{aligned} G_+(x, e^{i\alpha}) &= \begin{pmatrix} \cos\frac{\theta}{2} & -e^{-i\phi}\sin\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \exp(i\alpha\sigma_3)\ , \\ G_-(x, e^{i\alpha}) &= \begin{pmatrix} e^{-i\phi}\cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & e^{i\phi}\cos\frac{\theta}{2} \end{pmatrix} \exp(i\alpha\sigma_3)\ . \end{aligned} \tag{4.15}$$

This, indeed, gives a bundle morphism between P and $\tilde{P}$ as the given maps satisfy condition (4.12). Note that $G_+(x, 1)$ precisely corresponds to the singular gauge transformation γ in (4.10).

There is a well-known isomorphism between the total space S^3 of the Dirac monopole and the gauge group $\mathsf{SU}(2)$ of the 't Hooft–Polyakov monopole, which identifies the two complex numbers parameterizing S^3 with the diagonal and off-diagonal elements of a general $\mathsf{SU}(2)$-matrix. Under this isomorphism the above bundle morphism becomes trivial. To see this consider, e.g.,

$$\begin{aligned} G_+(x, e^{i\alpha}) &= \begin{pmatrix} \cos\frac{\theta}{2} & -e^{-i\phi}\sin\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \exp(i\alpha\sigma_3) \\ &= \begin{pmatrix} e^{i\alpha}\cos\frac{\theta}{2} & -e^{-i(\alpha+\phi)}\sin\frac{\theta}{2} \\ e^{i(\alpha+\phi)}\sin\frac{\theta}{2} & e^{-i\alpha}\cos\frac{\theta}{2} \end{pmatrix}\ . \end{aligned} \tag{4.16}$$

The local trivialization of the Hopf fibration on U_+ leads to

$$f_+^{-1}(x, e^{i\alpha}) = (e^{i\alpha}\cos\frac{\theta}{2}, e^{i(\alpha+\phi)}\sin\frac{\theta}{2})\ , \tag{4.17}$$

which agrees with the defining elements of the $\mathsf{SU}(2)$-matrix $G_+(x, e^{i\alpha})$. Thus, when identifying S^3 with $\mathsf{SU}(2)$ we have that G_+ is simply the identity and the bundle morphism is indeed trivial. We can summarize this in the following diagram

$$
\begin{tikzcd}
\mathsf{U}(1) \arrow[r, hook] \arrow[rr, bend left, dashed, "\varphi"] & \mathsf{SU}(2) \arrow[d, "\pi"] \arrow[rr, bend left, dashed, "\pi\times\mathrm{id}"] & \mathsf{SU}(2) \arrow[r, hook] & S^2 \times \mathsf{SU}(2) \arrow[d, "\mathrm{pr}_1"] \\
& S^2 \arrow[rr, bend left, dashed, "\mathrm{id}_{S^2}"] & & S^2
\end{tikzcd}\ , \tag{4.18}
$$

which will be our guiding principle in identifying a suitable gauge structure for the non-abelian self-dual string in the next section.

4.3 The abelian self-dual string and choosing a gauge structure

In string theory we have a system of D-branes corresponding to the monopoles above. More precisely, a monopole of charge m with gauge group $\mathsf{SU}(N)$ is given by a stack of m D1-branes ending on a stack of N D3-branes in type IIB superstring theory corresponding to the configuration [91] ,

$$
\begin{array}{cccccccc}
 & 0 & 1 & 2 & 3 & 4 & \dots \\
\mathrm{D1} & \times & & & & \times & \\
\mathrm{D3} & \times & \times & \times & \times & & & .
\end{array} \tag{4.19}
$$

T-dualizing to obtain a D2-D4 brane system and lifting to M-theory yields the configuration ,

$$
\begin{array}{cccccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & \dots \\
\mathrm{M2} & \times & & & & & \times & \times & \\
\mathrm{M5} & \times & \times & \times & \times & \times & \times & & & ,
\end{array} \tag{4.20}
$$

where we relabeled some of the coordinates for simplicity. This system is only well understood for a single M5-brane, which is the M-theoretic description of the abelian self-dual string. As mentioned above, this self-dual string is governed by the equation

$$\mathcal{H} = \mathrm{d}B = *\mathrm{d}\Phi \ . \tag{4.21}$$

This is a BPS solution of the equations of motion of the single M5-brane and, indeed, corresponds to a single M2-brane ending on an M5-brane [92]. It is defined on the $\mathbb{R}^4$-part of the worldvolume of the M5-brane that is disjoint from the M2-brane and, in this $\mathbb{R}^4$, the boundary of the M2-brane is a point $x_0 \in \mathbb{R}^4$. There is a further BPS solution to the M5-brane equations that corresponds to an M5-brane intersecting another M5-brane over a threebrane [98]. It would be interesting to also study a non-abelian version of this threebrane soliton, however, we will restrict ourselves to the self-dual string in this thesis.

Note that the self-dual string equation above can be seen as arising from a dimensional reduction of the self-duality equation $\mathcal{H} = *_6\mathcal{H}$ from $\mathbb{R}^{1,5}$ to $\mathbb{R}^4$ where the scalar field Φ is identified with the components of B along the reduced directions. Also, a further dimensional reduction to $\mathbb{R}^3$ yields the abelian Bogomolny monopole equations $\mathcal{F} = *_3\mathrm{d}\Phi$.

From the Bianchi identity $\mathrm{d}\mathcal{H} = 0$, we learn that

$$\Box\Phi = *\mathrm{d}\mathcal{H} = 0 \ , \tag{4.22}$$

that is, Φ is a harmonic function on $\mathbb{R}^4$. Therefore, interesting solutions will be singular at a point x_0. For a single self-dual string at x_0, the solution is [92]

$$\Phi = \frac{1}{(x - x_0)^2} \ , \tag{4.23}$$

and the concrete expression for the B-field, which is singular along lines going from the origin through opposite poles of S^3 to infinity, can be found (up to its radial dependence) e.g. in [99]. Because equation (4.21) is linear in both B and Φ, we can form linear combination of solutions to obtain new solutions. That is, the abelian self-dual strings do not interact.

This is fully analogous to the Dirac monopole and we can use this to find a suitable gauge structure for our non-abelian generalization. The 3-form curvature $\mathcal{H}$ of the abelian self-dual string is the curvature of the connective structure on an underlying abelian gerbe or principal $\mathsf{BU}(1)$-2-bundle over S^3, cf. Section 3.4. As such the abelian self-dual string corresponds to the principal G-2-bundle

$$\begin{tikzcd}
\mathsf{BU}(1) \arrow[r, hook] & \mathscr{G}_F \arrow[d, "\pi"] \\
 & (S^3 \leftleftarrows S^3)\ ,
\end{tikzcd} \tag{4.24}$$

similarly to (4.7).

In (4.18) we observed that in going from the Dirac monopole to the non-abelian 't Hooft–Polyakov monopole we used the total space P of the Dirac monopole as the manifold underlying the gauge group of the non-abelian monopole. To follow this analogy we want the total space $\mathscr{G}_F$ of the abelian self-dual string to serve as the gauge structure of our non-abelian generalization leading to the diagram

$$\begin{tikzcd}
\mathsf{BU}(1) \arrow[r, hook] \arrow[rr, bend left, dashed, "\varphi"] & \mathscr{G}_F \arrow[d, "\pi"] \arrow[rr, bend left, dashed, "\pi\times\mathrm{id}"] & \mathscr{G}_F \arrow[r, hook] & (S^3 \leftleftarrows S^3) \times \mathscr{G}_F \arrow[d, "\mathrm{pr}_1"] \\
 & (S^3 \leftleftarrows S^3) \arrow[rr, bend left, dashed, "\mathrm{id}"] & & (S^3 \leftleftarrows S^3)\ ,
\end{tikzcd} \tag{4.25}$$

where the maps are now morphisms of smooth 2-spaces.

For this picture to make sense, we need a Lie 2-group structure on the 2-space $\mathscr{G}_F$. This structure does exist and is given by the 2-group models of the string group of Section 2.6. There are different but equivalent ways of describing $\mathscr{G}_F$ as a bundle gerbe as in Definition 3.6 depending on what surjective submersion $\pi : Y \twoheadrightarrow \mathsf{SU}(2)$ is chosen.

Starting from an ordinary cover $Y_1 = \mathcal{U} = \sqcup_i U_i$ of $\mathsf{SU}(2)$ leads to the skeletal model of the string group, while the surjective submersion $Y_2 = P_0\mathsf{SU}(2) \twoheadrightarrow \mathsf{SU}(2)$ leads to the loop model. For our purposes we will only need the corresponding Lie 2-algebras, that is, the skeletal string algebra $\mathfrak{string}_{\mathrm{sk}}$ and the loop string algebra $\mathfrak{string}_{\hat{\Omega}}$. We introduced and defined these in Section 2.6 and, in the following, use them as a suitable candidate for the gauge structure of the non-abelian self-dual string.

4.4 Gauge covariance, categorical equivalence and fake flatness

Having decided on $\mathfrak{string}_{\rm sk}$ and $\mathfrak{string}_{\hat\Omega}$ as a suitable gauge structure we can now write down a non-abelian version of the self-dual string equation (4.2). From higher gauge theory we have expressions for the 3-form curvature $\mathcal{H}$, see equations (2.63) and (2.67). Thus, in the skeletal case, we have

$$\mathcal{H}_{\rm sk} = {\rm d}B + \tfrac{1}{6}\mu_3(A,A,A) = *{\rm d}\Phi\ , \tag{4.26}$$

and in the loop case

$$\mathcal{H}_{\hat\Omega} = {\rm d}B + \mu_2(A,B) = *{\rm d}\Phi\ , \tag{4.27}$$

as a natural generalization of (4.21). The relevant maps are defined in (2.62) and (2.66), respectively. Note that from the dimensional reduction of the self-duality equation $\mathcal{H} = *_6\mathcal{H}$ in six dimensions we also have that $\mu_1(\Phi) = 0$, so that in both cases Φ is an element of $\mathbb{R}$. However, to arrive at a full set of equations we still need a condition for two-form curvature $\mathcal{F}$, which we will investigate in the following.

In order for the above equations to be sensible they need to be gauge covariant. We obtain infinitesimal gauge transformations via homotopies which are partially flat, as detailed in Section 3.1. In the skeletal case this yields the transformations

$$\begin{aligned} \delta A &= {\rm d}\lambda + \mu_2(A,\lambda)\ , & \delta\mathcal{F} &= \mu_2(\mathcal{F},\lambda)\ ,\\ \delta B &= {\rm d}\Lambda - \tfrac{1}{2}\mu_3(A,A,\lambda)\ , & \delta\mathcal{H} &= \mu_3(\mathcal{F},A,\lambda)\ , \end{aligned} \tag{4.28}$$

which are parameterized by a 0-form λ taking values in $\mathfrak{g}$ and a 1-form Λ taking values in $\mathbb{R}$. Likewise, in the loop model we have

$$\begin{aligned} \delta A &= {\rm d}\lambda - \mu_1(\Lambda) + \mu_2(A,\lambda)\ , & \delta\mathcal{F} &= \mu_2(\mathcal{F},\lambda)\ ,\\ \delta B &= {\rm d}\Lambda + \mu_2(A,\Lambda) + \mu_2(B,\lambda)\ , & \delta\mathcal{H} &= \mu_2(\mathcal{H},\lambda) + \mu_2(\mathcal{F},\Lambda)\ , \end{aligned} \tag{4.29}$$

where now λ is a 0-form taking values in $P_0\mathfrak{g}$ and Λ is a 1-form taking values in $\Omega\mathfrak{g} \oplus \mathbb{R}$.

In both cases we recognize that, since Φ remains invariant under gauge transformations, the above non-abelian self-dual string equations are gauge covariant if and only if they are supplemented with the condition $\mathcal{F} = 0$. This condition is often called the **fake flatness condition** and appears frequently in higher gauge theory. It is, for example, responsible for rendering higher parallel transport of strings invariant under surface reparameterizations [2], see also [1, 32, 33].

Interestingly, this condition also emerges when considering categorical equivalences. While categorical equivalence is a very broad concept and physics is generally not invariant under it, one could hope that the self-dual string is equivalently expressible in the different models of the string algebra. In fact, we will see that this is indeed possible and one can freely move from the skeletal to the loop model and vice versa.

For now, however, a general morphism (ϕ_1, ϕ_2) of 2-term L_∞-algebras maps the potentials and curvatures of the self-dual string according to

$$A \mapsto \phi_1(A) \qquad \text{and} \qquad B \mapsto \phi_1(B) + \tfrac{1}{2}\phi_2(A, A)\ , \tag{4.30}$$

which then consequently leads to

$$\mathcal{F} \mapsto \phi_1(\mathcal{F}) \qquad \text{and} \qquad \mathcal{H} \mapsto \phi_1(\mathcal{H}) + \phi_2(\mathcal{F}, A)\ , \tag{4.31}$$

also cf. (2.28). For our case the explicit maps are given in (2.69) and (2.70). We also have $\Phi \mapsto \phi_1(\Phi)$ under this mapping. Thus, the transformation of $\mathcal{H}$ implies that in order for the non-abelian self-dual string to be equivalently expressible in both the skeletal and loop model the fake flatness condition again needs to be met.

This fake flatness condition, however, is too strong a condition for our purposes as it allows for gauge transformations to the abelian case, which we can readily see as follows.

In the skeletal case, fake flatness is just given by $\mathcal{F} = \mathrm{d}A + \frac{1}{2}\mu_2(A, A) = 0$, which as usual implies that locally A is pure gauge and, thus, can be gauge transformed to zero, see e.g. [80]. Then, equation (4.26) reduces to the abelian version and therefore $\mathcal{F} = 0$ cannot be part of the equations of motion of a truly non-abelian self-dual string.

The same statement is true in the loop space picture. Here, fake flatness is given by $\mathcal{F} = \mathrm{d}A + \frac{1}{2}\mu_2(A,A) + \mu_1(B) = 0$. From (4.29) one can see that this is a gauge invariant statement and for convenience, we gauge transform by $\Lambda = \chi(A)$ such that $A = f(\tau)\partial A =: f(\tau)A_0$ and $f(\tau) > 0$ for all $\tau \in [0,1]$. As before, ∂ is the end-point evaluation map and χ is defined as in (2.71). As $\mathcal{F} = 0$ we have that $\mathcal{F} - \mu_1(B)$ is in the image of μ_1 where

$$\mathcal{F} - \mu_1(B) = \mathrm{d}A + \tfrac{1}{2}\mu_2(A,A) = f(\tau)\mathrm{d}A_0 + \tfrac{1}{2}(f(\tau))^2\mu_2(A_0,A_0)\ . \tag{4.32}$$

This implies that $\mathrm{d}A_0 + \frac{1}{2}\mu_2(A_0,A_0) = 0$, so that A_0 is pure gauge. It follows that also A is of the form $A = g(\tau)\mathrm{d}(g(\tau))^{-1}$ and thus pure gauge. That is, once again, $\mathcal{F} = 0$ is too strong a condition to allow for non-abelian solutions.

This problem was in essence encountered before, in a general discussion of non-abelian higher gauge theories based on strict Lie 2-groups [100] as well as in a first approach of using higher gauge theory to describe non-abelian self-dual string [101].

The new loophole to alleviate this problem is to use the twisted versions of the string algebra models that we introduced in Section 3.5. Recall that the twisted string algebras are associated with a modified Weil algebra, which in turn changes the expressions for the higher curvatures. This suggests that the naive curvatures used in equations (4.26) and (4.27) should be replaced with the ones given in (3.49) and (3.53). This will lift the requirement of fake flatness for both gauge covariance and categorical equivalence, as we will now see.

Firstly, let us consider the infinitesimal gauge transformations. As the flat homotopies are now based on $\widehat{\mathfrak{string}}_{\mathrm{sk}}$ and $\widehat{\mathfrak{string}}_{\hat{\Omega}}$ these will also be modified. In the skeletal case the gauge transformations amount to

$$\begin{aligned}
&\delta A = \mathrm{d}\lambda + \mu_2(A,\lambda)\ , && \delta\mathcal{F} = \mu_2(\mathcal{F},\lambda)\ ,\\
&\delta B = \mathrm{d}\Lambda - \mu_1(\Sigma) + \kappa(\lambda,\mathcal{F}) - \tfrac{1}{2}\mu_3(A,A,\lambda)\ , && \delta\mathcal{H} = 0\ ,\\
&\delta C = \mathrm{d}\Sigma\ , && \delta\mathcal{G} = 0\ ,
\end{aligned} \tag{4.33}$$

which are, again, parameterized by a $\mathfrak{g}$-valued 0-form λ, a $\mathbb{R}$-valued 1-form Λ and a 2-form Σ valued in the additional copy of $\mathbb{R}$. Similarly, in the loop model we now have

$$
\begin{aligned}
\delta A &= \mathrm{d}\lambda + \mu_2(A,\lambda) - \mu_1(\Lambda)\ , & \delta C &= \mathrm{d}\Sigma\ ,\\
\delta B &= \mathrm{d}\Lambda - \mu_1(\Sigma) + \mu_2(A,\Lambda) - \mu_2(\lambda,B) + \kappa(\lambda,\mathcal{F})\ , & \delta\mathcal{H} &= 0\ ,\\
\delta\mathcal{F} &= \mu_2(\mathcal{F},\lambda) + \mu_1(\kappa(\lambda,\mathcal{F}))\ , & \delta\mathcal{G} &= 0\ .
\end{aligned}
\tag{4.34}
$$

Here, the gauge parameters are a $P_0\mathfrak{g}$-valued 0-form λ, a $\Omega\mathfrak{g}\oplus\mathbb{R}$-valued 1-form Λ and a $\mathbb{R}$-valued 2-form Σ. In both cases, the twist renders $\mathcal{H}$ gauge invariant, independent of any conditions on $\mathcal{F}$. Thus, we can write down gauge covariant self-dual string equations while allowing for a non-zero $\mathcal{F}$.

Secondly, we turn our attention to the issue of categorical equivalence between the different models of the string algebra. A morphism (ϕ_1,ϕ_2) of L_∞-algebras still maps the potentials according to (4.30), which again leads to $\mathcal{F}\mapsto\phi_1(\mathcal{F})$. The additional terms in $\mathcal{H}$, however, now modify the behavior of $\mathcal{H}$ under this mapping. More specifically, the extra terms cancel the appearance of $\phi_2(\mathcal{F},A)$ so that the 3-form curvature is simply mapped as $\mathcal{H}\mapsto\phi_1(\mathcal{H})$. To see this explicitly, consider the maps (ϕ_1,ϕ_2) and (ψ_1,ψ_2) as given in (2.69) and (2.70). With these we have

$$
\begin{aligned}
\kappa(\phi_1(A_{\mathrm{sk}}),\phi_1(\mathcal{F}_{\mathrm{sk}})) &= \big([A_{\mathrm{sk}},\mathcal{F}_{\mathrm{sk}}](f^2-f), 2(A_{\mathrm{sk}},\mathcal{F}_{\mathrm{sk}})\int_0^1 \dot{f} f\,\mathrm{d}\tau\big)\\
&= \big([\mathcal{F}_{\mathrm{sk}},A_{\mathrm{sk}}](f-f^2), 2(A_{\mathrm{sk}},\mathcal{F}_{\mathrm{sk}})\int_0^1 f\,\mathrm{d}f\big)\\
&= \big([\mathcal{F}_{\mathrm{sk}},A_{\mathrm{sk}}](f-f^2), (A_{\mathrm{sk}},\mathcal{F}_{\mathrm{sk}})\big)\\
&= \phi_2(\mathcal{F}_{\mathrm{sk}},A_{\mathrm{sk}}) + \phi_1(\kappa(A_{\mathrm{sk}},\mathcal{F}_{\mathrm{sk}}))\ ,
\end{aligned}
\tag{4.35}
$$

as well as

$$\begin{aligned}\psi_1(\kappa(A_{\hat{\Omega}},\mathcal{F}_{\hat{\Omega}})) &= 2\int_0^1 (\dot{A}_{\hat{\Omega}}, \mathrm{d}A_{\hat{\Omega}} + [A_{\hat{\Omega}}, A_{\hat{\Omega}}] + \mu_1(B_{\hat{\Omega}}))\,\mathrm{d}\tau \\ &= \int_0^1 2(\dot{A}_{\hat{\Omega}}, \mathrm{d}A_{\hat{\Omega}}) + \tfrac{1}{3}\frac{\partial}{\partial\tau}(A_{\hat{\Omega}}, [A_{\hat{\Omega}}, A_{\hat{\Omega}}]) - 2(A_{\hat{\Omega}}, \mu_1(\dot{B}_{\hat{\Omega}}))\,\mathrm{d}\tau \\ &= -\psi_2(\mathcal{F}_{\hat{\Omega}}, A_{\hat{\Omega}}) + \int_0^1 \frac{\partial}{\partial\tau}(A_{\hat{\Omega}}, \mathrm{d}A_{\hat{\Omega}}) + \tfrac{1}{2}\frac{\partial}{\partial\tau}(A_{\hat{\Omega}}, [A_{\hat{\Omega}}, A_{\hat{\Omega}}])\,\mathrm{d}\tau \\ &= -\psi_2(\mathcal{F}_{\hat{\Omega}}, A_{\hat{\Omega}}) + \kappa(\psi_1(A_{\hat{\Omega}}), \psi_1(\mathcal{F}_{\hat{\Omega}}))\ ,\end{aligned} \tag{4.36}$$

which, in conjunction with (4.31), shows that indeed now $\mathcal{H} \mapsto \phi_1(\mathcal{H})$ and $\mathcal{H} \mapsto \psi_1(\mathcal{H})$. Thus, again we are not required to impose the fake flatness condition and are able to have suitable, non-abelian self-dual string equations while having no constraints on $\mathcal{F}$. In fact, the above morphism will map gauge equivalence classes of solutions to gauge equivalence classes of solutions as desired, cf. Section 4.5.

In summary, using the twisted string algebras we are no longer required to impose the fake flatness condition. Being free to choose $\mathcal{F}$ we will instead introduce a natural equation for a non-zero $\mathcal{F}$ that allows for truly non-abelian solutions, which we will discuss in Section 4.5.

4.5 The self-dual string equations

After the discussion in Section 4.4, it is now straightforward to write down the 3-form part of the non-abelian self-dual string equation on $\mathbb{R}^4$ using the two twisted models of the string Lie 2-algebra. Note that as part of the twisted string algebras' gauge structure there is a 3-form potential C, which we will set to 0 for simplicity.

In the case of $\widehat{\mathfrak{string}}_{\mathrm{sk}}$, we then have kinematic data consisting of fields

$$A \in \Omega^1(\mathbb{R}^4) \otimes \mathfrak{g}\ , \quad B \in \Omega^2(\mathbb{R}^4) \otimes \mathbb{R}\ , \quad \Phi \in \Omega^0(\mathbb{R}^4) \otimes \mathbb{R}\ , \tag{4.37}$$

with curvatures as in (3.49) and gauge transformations as in (4.33). The natural analogue of the Bogomolny equation is now given by

$$\mathcal{H} := \mathrm{d}B - (A, \mathrm{d}A) - \tfrac{1}{3}(A, [A, A]) = *\mathrm{d}\Phi \ . \tag{4.38}$$

In order to identify a suitable equation for $\mathcal{F}$, recall that the Bianchi identity leads to

$$* \, \mathrm{d}\mathcal{H} = - * (\mathcal{F}, \mathcal{F}) = \Box\Phi \ , \tag{4.39}$$

and therefore the Higgs field Φ is determined by the second Chern class which captures instantons on $\mathbb{R}^4$. Since knowing the Higgs field should suffice to describe the self-dual string modulo gauge invariance, it is natural to replace fake flatness $\mathcal{F} = 0$ with the instanton equation

$$\mathcal{F} = *\mathcal{F} \ . \tag{4.40}$$

This result is also in agreement with a different point of view. In the six-dimensional $\mathcal{N} = (1, 0)$ supersymmetric model of [47] and Chapter 5, the BPS equation leads to $\Box\Phi = - * (\mathcal{F}, *\mathcal{F})$ [102]. This BPS equation follows from our equation (4.39), if it is supplemented with the instanton equation $\mathcal{F} = *\mathcal{F}$. For a discussion of the full implications of our choice of gauge structure for this (1,0)-model see Chapter 5.

As a first consistency check, note that by putting $A = 0$, our equations (4.38) and (4.40) reduce to the abelian self-dual string equation (4.2).

Another consistency check that we can immediately perform is the reduction from M2-branes ending on M5-branes to D2-branes ending on D4-branes. That is, we dimensionally reduce $\mathbb{R}^4$ along an M-theory direction, say x^4. The resulting kinematical data consists of the following fields

$$\begin{aligned} &\breve{A}_1 \in \Omega^1(\mathbb{R}^3) \otimes \mathfrak{g} \ , \qquad \breve{\Phi}_1 \in \Omega^0(\mathbb{R}^3) \otimes \mathbb{R} \ , \qquad \breve{B} \in \Omega^2(\mathbb{R}^3) \otimes \mathbb{R} \ , \\ &\breve{A}_2 \in \Omega^1(\mathbb{R}^3) \otimes \mathbb{R} \ , \qquad \breve{\Phi}_2 \in \Omega^0(\mathbb{R}^3) \otimes \mathfrak{g} \ . \end{aligned} \tag{4.41}$$

Our equations (4.38) and (4.40) reduce to the following expressions:

$$
\begin{aligned}
*_3\nabla\breve{\Phi}_2 &= \mathrm{d}\breve{A}_1 + \tfrac{1}{2}\mu_2(\breve{A}_1,\breve{A}_1) =: \breve{\mathcal{F}}\ ,\\
0 &= \mathrm{d}\breve{B} - (\breve{A}_1,\mathrm{d}\breve{A}_1) - \tfrac{1}{3}(\breve{A}_1,[\breve{A}_1,\breve{A}_1]) =: \breve{\mathcal{H}}_1\ ,\\
*_3\mathrm{d}\breve{\Phi}_1 &= \mathrm{d}\breve{A}_2 - (\breve{A}_1,\mathrm{d}\breve{\Phi}_2) - (\breve{\Phi}_2,\mathrm{d}\breve{A}_1) - (\breve{A}_1,[\breve{A}_1,\breve{\Phi}_2]) =: \breve{\mathcal{H}}_2\ .
\end{aligned}
\tag{4.42}
$$

Here the first equation is just the monopole equation on $\mathbb{R}^3$ for connection $\breve{A}_1$ and Higgs field $\breve{\Phi}_2$. The second equation can be satisfied by choosing an appropriate $\breve{B}$. This is possible by Poincaré's lemma, since the form to be canceled by $\mathrm{d}\breve{B}$ is a top form on $\mathbb{R}^3$, hence closed. The third equation can be rewritten as

$$
_3\,\mathrm{d}\breve{\Phi}_1 - \mathrm{d}\breve{A}_2 = -2(\breve{\Phi}_2,\breve{\mathcal{F}}) + \mathrm{d}(\breve{\Phi}_2,\breve{A}_1) = -\mathrm{d}(\breve{\Phi}_2,\breve{\Phi}_2) + \mathrm{d}(\breve{\Phi}_2,\breve{A}_1)\ , \tag{4.43}
$$

where we used $\breve{\mathcal{F}} = *_3\nabla\breve{\Phi}_2$. This is clearly solved by

$$
\breve{A}_2 = -(\breve{\Phi}_2,\breve{A}_1) \qquad \text{and} \qquad \breve{\Phi}_1 = -(\breve{\Phi}_2,\breve{\Phi}_2)\ . \tag{4.44}
$$

Altogether, the dimensional reduction of our self-dual string equations given in (4.38) and (4.40) leads to the Bogomolny monopole equations on $\mathbb{R}^3$, as expected from string theory.

Next, let us consider the corresponding equations for the model $\widehat{\mathfrak{string}}_{\hat{\Omega}}$. Here, the kinematic data is given by fields

$$
A \in \Omega^1(\mathbb{R}^4)\otimes P_0\mathfrak{g}\ ,\quad B\in\Omega^2(\mathbb{R}^4)\otimes(\Omega\mathfrak{g}\oplus\mathbb{R})\ ,\quad \Phi\in\Omega^0(\mathbb{R}^4)\otimes(\Omega\mathfrak{g}\oplus\mathbb{R})\ , \tag{4.45}
$$

with curvatures as in (3.53) and gauge transformations as in (4.34). A suitable set of equations for the self-dual string are now given by

$$
\begin{aligned}
&H := \mathrm{d}B + \mu_2(A,B) - \kappa(A,\mathcal{F}) = *\nabla\Phi\ ,\\
&\mathcal{F} = *\mathcal{F}\ ,\quad \mu_1(\Phi) = 0\ .
\end{aligned}
\tag{4.46}
$$

Note that the third equation[2] arises in the dimensional reduction of the self-dual

[2]This condition is trivially satisfied in the skeletal case.

equation $\mathcal{H} = *_6\mathcal{H}$ in six dimensions, as mentioned above.

The dimensional reduction to monopoles is now accomplished by restricting to the endpoint in path space, $\partial\mathcal{F}$, and projecting onto $\mathbb{R}$ in $\Omega\mathfrak{g}\oplus\mathbb{R}$, where we recover the skeletal situation.

Furthermore, it is not hard to see that in the categorical equivalence of equations (4.46) to (4.38) and (4.40) we have that gauge equivalence classes of solutions to each of these equations are in one-to-one correspondence with gauge equivalence classes of solutions of the respective other ones. The correspondence is established, as before, by the maps (2.69) and (2.70) of the underlying Lie 2-algebras

Explicitly, given a solution $(A_{\rm sk}, B_{\rm sk}, \Phi_{\rm sk})$ to (4.38) and (4.40), one readily verifies that

$$A_{\hat\Omega} = A_{\rm sk} f(\tau)~,\quad B_{\hat\Omega} = (\tfrac12[A_{\rm sk},A_{\rm sk}](f(\tau)-f^2(\tau)),B_{\rm sk})\qquad\text{and}\qquad \Phi_{\hat\Omega} = (0,\Phi_{\rm sk}) \tag{4.47}$$

is a solution to (4.46). Gauge transformations of $(A_{\rm sk}, B_{\rm sk}, \Phi_{\rm sk})$ parameterized by $(\lambda_{\rm sk},\Lambda_{\rm sk})$ are mapped to gauge transformations of $(A_{\hat\Omega}, B_{\hat\Omega}, \Phi_{\hat\Omega})$ parameterized by

$$\lambda_{\hat\Omega} = \lambda_{\rm sk} f(\tau)\qquad\text{and}\qquad \Lambda_{\hat\Omega} = \big([\lambda_{\rm sk},A_{\rm sk}](f(\tau)-f^2(\tau))~,~\Lambda_{\rm sk}\big)~. \tag{4.48}$$

Conversely given a solution $(A_{\hat\Omega}, B_{\hat\Omega}, \Phi_{\hat\Omega})$, it is straightforward to check that

$$A_{\rm sk} = \partial A_{\hat\Omega}~,\quad B_{\rm sk} = \mathsf{pr}_{\mathbb{R}} B_{\hat\Omega} + \int_0^1 \mathrm{d}\tau~(A_{\hat\Omega},\dot A_{\hat\Omega})~,\quad \Phi_{\rm sk} = \mathsf{pr}_{\mathbb{R}}\Phi_{\hat\Omega} \tag{4.49}$$

is a solution to (4.38) and (4.40). Moreover, gauge transformations parameterized by $(\lambda_{\hat\Omega},\Lambda_{\hat\Omega})$ are mapped to gauge transformations parameterized by

$$\lambda_{\rm sk} = \partial\lambda_{\hat\Omega}\qquad\text{and}\qquad \Lambda_{\rm sk} = \mathsf{pr}_{\mathbb{R}}\Lambda_{\hat\Omega} + \int_0^1 \mathrm{d}\tau~(\dot\lambda_{\hat\Omega},A_{\hat\Omega}) - (\lambda_{\hat\Omega},\dot A_{\hat\Omega})~. \tag{4.50}$$

4.6 Bogomolny bound

Recall that both the instanton and monopole equations can be derived as equations for the Bogomolny bound of a suitable action principle, cf. Section 4.2. The same is true for our non-abelian self-dual string equations.

For simplicity, we restrict ourselves to the skeletal case $\widehat{\mathfrak{string}}_{\rm sk}$. We then have the following obvious action functional of higher Yang–Mills–Higgs theory:

$$S = \int_{\mathbb{R}^4} \mathcal{H} \wedge *\mathcal{H} + \mathrm{d}\Phi \wedge *\mathrm{d}\Phi + (\mathcal{F}, *\mathcal{F}) \ , \tag{4.51}$$

where $\mathcal{F}$, $\mathcal{H}$ and Φ are the 2- and 3-form curvature as well as the Higgs field introduced in the previous sections. For $\Phi = 0$, this action was given before in [3] in a more general context, where it was, however, not gauge invariant. Since $\delta\mathcal{H} = 0$ for the skeletal string algebra (as well as for the twisted loop string algebra), this problem does not arise here. We can recast this action in the following form:

$$S = \int_{\mathbb{R}^4} (\mathcal{H}-*\mathrm{d}\Phi)\wedge*(\mathcal{H}-*\mathrm{d}\Phi) - 2\mathcal{H}\wedge\mathrm{d}\Phi + \tfrac{1}{2}\big((\mathcal{F}-*\mathcal{F}), *(\mathcal{F}-*\mathcal{F})\big) + \big(\mathcal{F},\mathcal{F}\big) \ . \tag{4.52}$$

As expected, the minimum of this action is given by solutions to our self-dual string equations

$$\mathcal{H} = *\mathrm{d}\Phi \ , \quad \mathcal{F} = *\mathcal{F} \ , \tag{4.53}$$

and for such solutions, the action is given by the bound

$$S = -2\int_{\mathbb{R}^4} \mathcal{H} \wedge \mathrm{d}\Phi + \int_{\mathbb{R}^4} \big(\mathcal{F},\mathcal{F}\big) = -2\int_{\mathbb{R}^4} \mathcal{H} \wedge \mathrm{d}\Phi + \int_{S^3_\infty} \mathcal{H} \ , \tag{4.54}$$

where we used the Bianchi identity $\mathrm{d}\mathcal{H} = (\mathcal{F},\mathcal{F})$.

4.7 The elementary solution

Let us now come to the explicit form of the elementary solution. In analogy with the 't Hooft–Polyakov monopole we choose $\mathfrak{g} = \mathfrak{su}(2)$. Starting with the case of the skeletal algebra $\widehat{\mathfrak{string}}_{\rm sk}(\mathfrak{su}(2))$, the relevant field content is (4.37) and we wish to solve

$$\mathcal{H} = \mathrm{d}B - (A, \mathrm{d}A) - \tfrac{1}{3}(A, [A, A]) = *\mathrm{d}\Phi \qquad \text{and} \qquad \mathcal{F} = \mathrm{d}A + \tfrac{1}{2}[A, A] = *\mathcal{F} \ . \tag{4.55}$$

We start from the elementary instanton solution and a trivial 2-form potential,

$$A_\mu(x) = -\mathrm{i}\frac{\eta^i_{\mu\nu}\,\sigma_i\,(x^\nu - x^\nu_0)}{\rho^2 + (x - x_0)^2}~, \quad B(x) = 0~, \tag{4.56}$$

where σ_i are the Pauli matrices satisfying $[\sigma_i, \sigma_j] = 2\mathrm{i}\varepsilon_{ijk}\sigma_k$ and $\eta^i_{\nu\kappa}$ are the 't Hooft symbols, which allow to form self-dual 2-forms from the generators of $\mathfrak{su}(2)$. These are defined for $i = 1, 2, 3$ and $\mu, \nu = 1, 2, 3, 4$ and are given by

$$\eta^i_{\mu\nu} = \begin{cases} \varepsilon_{i\mu\nu}~, & \mu, \nu = 1, 2, 3~, \\ \delta_{i\mu}~, & \nu = 4~, \\ -\delta_{i\nu}~, & \mu = 4~, \\ 0~, & \mu = \nu = 4~. \end{cases} \tag{4.57}$$

The variables $x_0 \in \mathbb{R}^4$ and $\rho \in \mathbb{R}$ denote the position and the size of the elementary instanton. We normalize the inner product $(-,-)$ on $\mathfrak{su}(2)$ such that we have

$$(x, y) = \mathrm{tr}(x^\dagger y) \qquad \text{with} \qquad (\mathrm{i}\sigma_i, \mathrm{i}\sigma_j) = (\sigma_i, \sigma_j) = \delta_{ij}~. \tag{4.58}$$

With these conventions, we find that

$$\Phi(x) = \frac{(x - x_0)^2 + 2\rho^2}{\big((x - x_0)^2 + \rho^2\big)^2} \tag{4.59}$$

completes the solution.

Let us now perform the obvious consistency checks on the solution given in (4.56) and (4.59). First of all, it is evident that this solution is non-singular on all of $\mathbb{R}^4$, which sets it apart from the abelian solution (4.23). Furthermore, it is interacting in the sense that non-trivial linear combinations of this solution are no longer solutions. This clearly shows that our result is not an abelian solution simply recast in an unusual form, but rather a genuinely non-abelian self-dual string. Additionally, in the limit $|x| \to \infty$, we have that, $\Phi \sim \frac{1}{x^2}$, which is the solution to the abelian self-dual string, as expected.

The moduli of our elementary solution are the same as those of the instanton: the position x_0, the size parameter ρ as well as a global gauge transformation $g \in \mathsf{SU}(2)$.

The size parameter is the Goldstone mode arising from the break down of conformal invariance of the instanton equation $\mathcal{F} = *\mathcal{F}$ by choosing a specific solution (4.56).

For the loop space model $\widehat{\mathfrak{string}}_{\hat{\Omega}}(\mathfrak{su}(2))$, we can simply use the categorical equivalence to translate our solution. Here, the relevant field content is listed in (4.45) with the corresponding curvatures (3.53) and equations of motion (4.46).

Using (4.47) on the skeletal solution (4.56), we straightforwardly obtain the potentials

$$\begin{aligned} A_\mu(x) &= -\mathrm{i}\frac{\eta^i_{\mu\nu}\,\sigma_i\,(x^\nu - x^\nu_0)}{\rho^2 + (x-x_0)^2} f(\tau)\ , \\ B_{\mu\nu}(x) &= \left(-2\,\mathrm{i}\,\epsilon_{ijk}\,\sigma_k \frac{\eta^i_{\mu\kappa}\,(x-x_0)^\kappa\,\eta^j_{\nu\lambda}\,(x-x_0)^\lambda}{(\rho^2+(x-x_0)^2)^2}(f(\tau)-f^2(\tau))\ ,\ 0\right)\ , \end{aligned} \tag{4.60}$$

as well as the Higgs field

$$\Phi(x) = \left(0, \frac{(x-x_0)^2 + 2\rho^2}{\big((x-x_0)^2+\rho^2\big)^2}\right)\ . \tag{4.61}$$

These indeed form a solution to equations (4.46), as expected. Conversely, we recover the skeletal case from the inverse morphism of gauge potentials.

4.8 The global picture: string structures

While our discussion so far is consistent on flat $\mathbb{R}^4$, the full geometric picture has a remaining issue if one wants the gauge structure to lift to a $\mathfrak{string}$-connection, cf. Section 3.5. This corresponds to what is called a string structure as defined in [61, 77, 103–105]. For simplicity, we shall discuss this problem for the skeletal string algebra; the corresponding discussion in the loop case follows directly.

The fact that we consider instantons on $\mathbb{R}^4$ suggests that we should be working on a compactification M of $\mathbb{R}^4$ such as S^4. In this case, the first fractional Pontryagin class $\frac{1}{2}p_1 = (\mathcal{F}, \mathcal{F})$ is not trivial in $H^4(M, \mathbb{Z})$. This, however, would be a requirement for our gauge potentials to live on a principal 2-bundle corresponding to a string structure, again cf. the discussion in Section 3.5.

There is a rather obvious loophole to this problem. We can extend the structure

L_∞-algebra $\widehat{\mathfrak{string}}_{\rm sk}(\mathfrak{su}(2))$ in such a way that the additional degrees of freedom compensate the instanton contribution to the first Pontryagin class. A rather natural solution is to replace this algebra by $\widehat{\mathfrak{string}}_{\rm sk}(\mathfrak{su}(2)_L \times \mathfrak{su}(2)_R)$. This also brings our equations closer to the M2-brane models of [40–42].

The Pontryagin classes with respect to both $\mathfrak{su}(2)$-factors are additive, and we can compensate the instanton $\mathcal{F}_L$ in the left factor $\mathfrak{su}(2)_L$ with an anti-instanton $\mathcal{F}_R$ in the right factor $\mathfrak{su}(2)_R$ to obtain $[\frac{1}{2}p_1] = 0$ in $H^4(M, \mathbb{Z})$. Altogether, we arrive at the equations

$$\begin{gathered} \mathcal{F}_L = *\mathcal{F}_L~, \quad \mathcal{F}_R = - * \mathcal{F}_R~, \quad \mathcal{H} = *\mathrm{d}\Phi~, \\ [\tfrac{1}{2}p_1] = [(\mathcal{F}, \mathcal{F})] = [(\mathcal{F}_L, \mathcal{F}_L) + (\mathcal{F}_R, \mathcal{F}_R)] = 0~. \end{gathered} \tag{4.62}$$

We note that an alternative way of arriving at equivalent data is to flip the sign of the Killing form on $\mathfrak{su}_R(2)$, leading to an indefinite metric on $\mathfrak{su}_L(2) \oplus \mathfrak{su}_R(2)$, which is precisely the gauge algebra underlying the simplest M2-brane model. In this case, both $\mathcal{F}_L$ and $\mathcal{F}_R$ are chosen self-dual.

From our previous results in section 4.7, we readily glean the following extended solution:

$$A_{\mu,L}(x) = -\mathrm{i}\frac{\eta^i_{\mu\nu}\,\sigma_i\,(x^\nu - x^\nu_{0,L})}{\rho_L^2 + (x - x_{0,L})^2}~, \quad A_{\mu,R}(x) = -\mathrm{i}\frac{\bar{\eta}^i_{\mu\nu}\,\sigma_i\,(x^\nu - x^\nu_{0,R})}{\rho_R^2 + (x - x_{0,R})^2}~, \tag{4.63}$$

$$B(x) = 0~, \quad \Phi = \frac{(x - x_{0,L})^2 + 2\rho_L^2}{\big((x - x_{0,L})^2 + \rho_L^2\big)^2} - \frac{(x - x_{0,R})^2 + 2\rho_R^2}{\big((x - x_{0,R})^2 + \rho_R^2\big)^2}~, \tag{4.64}$$

where the 't Hooft tensors $\bar{\eta}^i_{\mu\nu}$ are defined analogously to the 't Hooft tensors in (4.57) and form a basis for anti-self-dual 2-forms in four dimensions. Note that the instanton and the anti-instanton do not need to have the same size ρ, nor do they have to be centered at the same point x_0. If all the moduli agree, then evidently $\Phi = 0$ and thus $\mathcal{H} = 0$.

The above data on $\mathbb{R}^4 = S^4 \backslash \{\infty\}$ provides us now with a non-trivial and well-defined string structure on S^4. We have a principal $\mathfrak{string}(\mathfrak{su}(2) \times \mathfrak{su}(2))$-bundle with connection defined by $A = A_L + A_R$ and the projections onto the two underlying $\mathsf{SU}(2)$-bundles are clearly topologically non-trivial: carrying an instanton and an anti-instanton, their individual Pontryagin classes do not vanish. A further characteristic class relevant here is $\mathcal{H}$. This should be understood as the Dixmier–Douady class $\mathrm{d}B$ of an abelian gerbe (which, by itself, is necessarily trivial on S^4) together with a coboundary $\mathrm{cs}(A)$ that trivializes a 2-gerbe with 4-form curvature $(\mathcal{F}, \mathcal{F})$. Note that $\mathcal{H}$ is gauge invariant and therefore isomorphisms of principal 2-bundles (which are gauge transformations) will not affect its value. It is therefore indeed an invariant of string structures.

Chapter 5

A 6D (1,0) Superconformal Model

In this chapter, we present the work done in [50], which uses the formalism detailed in the previous chapters to combine the constructions of [47, 55, 56] into an interesting $\mathcal{N} = (1, 0)$ superconformal action. We start with an outline of the ingredients before we discuss the action, its supersymmetries and the equations of motion.

5.1 Context

In [50] we endeavor to construct a six-dimensional $\mathcal{N} = (1, 0)$ superconformal model that contains the full field content of the (2,0)-theory and satisfies many of the items on the wishlist mentioned in the introduction. The non-abelian self-dual strings discussed above should be BPS states of this theory and, therefore, the gauge structure chosen in Chapter 4 should be an interesting candidate for the gauge structure of our model.

Our starting point is the $\mathcal{N} = (1, 0)$ superconformal model presented in [47], which we simply call the **(1,0)-model**. The (1,0)-model was obtained by writing down an ansatz for suitable supersymmetry transformations for a non-abelian tensor multiplet and deriving algebraic and dynamic conditions for their closure. This is the same method that led to the BLG M2-brane model [40, 41].

The gauge structure of the (1,0)-model and its gauge field contents were derived from non-abelian tensor hierarchy, see e.g. [106] and references in [47]. Maximal supergravities can be constructed by compactifying 10- and 11-dimensional supergravities on a torus. Each maximal supergravity exhibits a duality group G, and the theory can be deformed by rendering the one-form gauge potentials non-abelian with the gauge group being a subgroup of the duality group G. The precise structure is encoded in a representation of the subgroup of G and the embedding tensor Θ. The algebraic structure yields covariant derivatives and field strengths which, however, may not transform covariantly. In such cases, one is led to introduce a compensating non-abelian 2-form gauge potential, whose 3-form curvature may also not transform covariantly. This, in turn, forces the introduction of a compensating non-abelian 3-form potential and so on.

In total, the gauge structure of the (1,0)-model of [47] is then encoded by five structure constants g^{Ir}, h_I^r, b_{Irs}, d_{rs}^I and f_{rs}^t, where the various indices label the different spaces in which the potentials and curvatures take values. The curvatures can be expressed using these structure constants and the requirement that these transform covariantly leads to the following set of constraints

$$\begin{aligned} 2(d^J_{r(u}d^I_{v)s} - d^I_{rs}d^J_{uv})h^s_J &= 2f^s_{r(u}d^I_{v)s} - b_{Jsr}d^J_{uv}g^{Is} \ , \\ (d^J_{rs}b_{Iut} + d^J_{rt}b_{Isu} + 2d^K_{ru}b_{Kst}\delta^J_I)h^u_J &= f^u_{rs}b_{Iut} + f^u_{rt}b_{Isu} + g^{Ju}b_{Iur}b_{Jst} \ , \end{aligned} \tag{5.1}$$

which are linear in f, g and h. Furthermore, we obtain constraints that are bilinear in f, g and h given by

$$\begin{aligned} f^u_{[pq}f^s_{r]u} - \tfrac{1}{3}h^s_I d^I_{u[p}f^u_{qr]} &= 0 \ , \\ h^r_I g^{Is} &= 0 \ , \\ f^t_{rs}h^r_I - d^J_{rs}h^t_J h^r_I &= 0 \ , \\ g^{Js}h^r_K b_{Isr} - 2h^s_I h^r_K d^J_{rs} &= 0 \ , \\ -f^s_{rt}g^{It} + d^J_{rt}h^s_J g^{It} - g^{It}g^{Js}b_{Jtr} &= 0 \ . \end{aligned} \tag{5.2}$$

Finally, in order to write down action principles we have to impose the further constraints

$$b_{Jr(s}d^J_{uv)} = 0 \ , \quad h^r_I = \eta_{IJ}g^{Jr} \qquad \text{and} \qquad d^I_{rs} = \tfrac{1}{2}\eta^{IJ}b_{Jrs} \ , \tag{5.3}$$

where η_{IJ} is an additional constant, non-degenerate tensor describing a metric on the gauge algebra.

As shown in [52], these structure constants together with the above constraints underlying the non-abelian tensor hierarchy can be interpreted as L_∞-algebras with cyclic structure and the iterative construction of higher form potentials leads to formulas reminiscent of the ones for the curvatures, gauge transformations and Bianchi identities as found in Chapter 3, see also [107] and [108]. In particular, the skeletal model of the twisted string algebra presented in Section 3.5 together with a cyclic structure yields a suitable L_∞-algebra and can serve as a basis for this theory. More precisely, the expressions for the 2-form and 3-form curvatures $\mathcal{F}$ and $\mathcal{H}$ in (3.49) together with a cyclic structure lead to the same expressions as in the (1,0)-model. However, the same procedure as done in Section 3.5 for the higher curvatures leads to discrepancies, also cf. Section 5.7. We will postpone these considerations and, here, adopt the (1,0)-model as a useful starting point for our constructions.

To achieve our goal of constructing a model satisfying many of the items of the wish-list in the introduction, we have to address a few issues with the action of the (1,0)-model. First, we have to complete the field content to contain that of the full $(2,0)$-theory, even though we are just looking for an $\mathcal{N} = (1,0)$ superconformal action. This is analogous to the ABJM model, which is only $\mathcal{N} = 6$ supersymmetric, but has the same field content as the full $\mathcal{N} = 8$ BLG model. For this, we can rely on the results of [55]. Second, we would like to incorporate the PST formalism [53, 54] to include self-duality of the 3-form curvature as an equation of motion of the action. For the bosonic part of the $(1,0)$-model, this has been done in [56]. An extension to the supersymmetric case was announced, but this has not appeared yet in the literature. With our gauge structure this extension is indeed possible. As the details of our PST mechanism differ in some aspects from those of [56], it is not clear if such a construction is possible in the general gauge structure of [47].

Furthermore, choosing the twisted skeletal string algebra as the gauge structure directly solves some of the issues with the (1,0)-model and its PST extension: the nature of the gauge structure is clarified, the cubic interactions vanish and the PST and hypermultiplet extensions are rather straightforward.

We discussed further properties of the resulting model in Section 1.3 and comment on remaining issues in the conclusions.

5.2 The higher gauge algebra

We start by giving the higher gauge algebra that we use as the gauge structure for our model. Following the outline of the previous section, we choose to base this gauge structure on the skeletal model of the string algebra $\widehat{\mathfrak{string}}_{\rm sk}(\mathfrak{g})$. Note that the twisted 3-form curvature $\mathcal{H}$ in (3.49) is in agreement with the expression derived from tensor hierarchy in [47]. However, in order to write down an action principle we need an inner product on $\widehat{\mathfrak{string}}_{\rm sk}(\mathfrak{g})$ corresponding to the metric tensor η_{IJ} as in (5.3).

As discussed in Section 2.7 such an inner product arises naturally from a cyclic structure. As $\widehat{\mathfrak{string}}_{\rm sk}(\mathfrak{g})$ does not carry such a cyclic structure we minimally extend the algebra as before. That is, we double the algebra and arrive at

$$\begin{array}{ccccc} \mathbb{R}^*_q & \xleftarrow{\ \mathrm{id}^*\ } & \mathbb{R}^*_s[1] & & \mathfrak{g}^*[2] \\ \oplus & & \oplus & & \oplus \\ \mathfrak{g} & & \mathbb{R}_r[1] & \xleftarrow{\ \mathrm{id}\ } & \mathbb{R}_p[2]\ , \end{array} \tag{5.4}$$

with coordinates (x^α, q) of degree 0, (r,s) of degree 1 and (p, y_α) of degree 2. On the corresponding Q-manifold we then have the natural symplectic structure given by

$$\omega = \mathrm{d}x^\alpha \wedge \mathrm{d}y_\alpha + \mathrm{d}r \wedge \mathrm{d}s + \mathrm{d}p \wedge \mathrm{d}q\ , \tag{5.5}$$

cf. (2.83). As in (2.88), this yields the cyclic inner product on $\widehat{\mathfrak{string}}_{\rm sk}(\mathfrak{g})$

$$\begin{aligned} &\langle x_1 + q_1 + r_1 + s_1 + p_1 + y_1, x_2 + q_2 + r_2 + s_2 + p_2 + y_2 \rangle \\ &= y_1(x_2) + y_2(x_1) + p_1 q_2 + q_1 p_2 + r_1 s_2 + s_1 r_2\ , \end{aligned} \tag{5.6}$$

which provides the necessary structure to define a metric η_{IJ} as in (5.3).

As before, we also minimally extend the homological vector field of the corresponding Q-manifold to form an L_∞-algebra on the total space, which induces additional dualized brackets μ_i. For convenience, we again list the resulting maps, which are given by

$$
\begin{aligned}
&\mu_1 : \mathbb{R}^*_s[1] \to \mathbb{R}^*_q : \ \mu_1(s) := s \ , \\
&\mu_1 : \mathbb{R}_p[2] \to \mathbb{R}_r[1] : \ \mu_1(p) := p \ , \\
&\mu_2 : \mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{g} : \ \mu_2(x_1, x_2) := [x_1, x_2] \ , \\
&\mu_2 : \mathfrak{g} \wedge \mathfrak{g}^*[2] \to \mathfrak{g}^*[2] : \ \mu_2(x, y) := y([x, -]) \ , \\
&\mu_3 : \mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g} \to \mathbb{R}_r[1] : \ \mu_3(x_1, x_2, x_3) := (x_1, [x_2, x_3]) \ , \\
&\mu_3 : \mathfrak{g} \wedge \mathfrak{g} \wedge \mathbb{R}^*_s[1] \to \mathfrak{g}^*[2] : \ \mu_3(x_1, x_2, s) := \langle (-, [x_1, x_2]), s \rangle \ .
\end{aligned}
\tag{5.7}
$$

As we are now working with the twisted string algebra, we also have the Killing form κ, which together with its dualized version refines the 3-term L_∞-algebra structure:

$$
\begin{aligned}
&\kappa : \ \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}_r[1] : \kappa(x_1, x_2) := (x_1, x_2) \ , \\
&\kappa : \ \mathfrak{g} \otimes \mathbb{R}^*_s[1] \to \mathfrak{g}^*[2] : \kappa(x, s) := 2\langle \kappa(-, x), s \rangle \ .
\end{aligned}
\tag{5.8}
$$

The factor of 2 is introduced to better match the conventions of [47], but is not essential. For all other arguments, the maps μ_i and κ vanish.

This 3-term L_∞-algebra is sufficient to write down an action. In order to encode the full duality relations for differential forms in six dimensions, however, we have to add a non-propagating four-form. We therefore have to extend this 3-term L_∞-algebra trivially to a 4-term L_∞-algebra as follows:

$$
\begin{array}{ccccccc}
\mathbb{R}^*_q & \xleftarrow{\ \mathrm{id}^*\ } & \mathbb{R}^*_s[1] & & \mathfrak{g}^*[2] & \xleftarrow{\ \mathrm{id}\ } & \mathfrak{g}^*[3] \\
\oplus & & \oplus & & \oplus & & \\
\mathfrak{g} & & \mathbb{R}_r[1] & \xleftarrow{\ \mathrm{id}\ } & \mathbb{R}_p[2] & &
\end{array} \ ,
\tag{5.9}
$$

where we introduce another copy of $\mathfrak{g}^*$ in degree 3 with coordinate z_α. This is the diagram to have in mind when we will discuss the field content and the action below.

The only additional maps we define are

$$
\begin{aligned}
&\mu_1 : \mathfrak{g}^*[3] \to \mathfrak{g}^*[2] : \ \mu_1(z) := z \ , \\
&\mu_2 : \mathfrak{g} \wedge \mathfrak{g}^*[3] \to \mathfrak{g}^*[3] : \ \mu_2(x,z) := z([x,-]) \ , \\
&\kappa : \mathfrak{g} \otimes \mathfrak{g}^*[2] \to \mathfrak{g}^*[3] : \ \kappa(x,z) := z([x,-]) \ .
\end{aligned} \tag{5.10}
$$

We denote the resulting extended 4-term L_∞-algebra simply by $\widehat{\mathfrak{string}}_{\mathrm{ext}}(\mathfrak{g})$. Note that in adding $\mathfrak{g}^*[3]$ the whole algebra $\widehat{\mathfrak{string}}_{\mathrm{ext}}(\mathfrak{g})$ is again isomorphic to $\mathfrak{g}$ in cohomology. In principle, one would like to repeat the calculation of Section 3.5 in order to find the twisted expressions of the higher curvatures. However, we will use the curvatures as given in [47], which for $\mathcal{H}$ agree with the twist of the smaller algebra as given in Section 3.5, cf. the discussion in the next section and the remarks in Section 5.7.

The maps above indeed encode the structure constants of a (1,0)-model as introduced above. The explicit dictionary to the structure constants used in [47] is the following:

	In [47, 56]	Translated to $\widehat{\mathfrak{string}}(\mathfrak{g})$
Indices	T^r	$T^\alpha + T_q \in \mathfrak{g} \oplus \mathbb{R}^*_q$
(T: general obj.)	T^I	$T_r + T_s \in \mathbb{R}_r[1] \oplus \mathbb{R}^*_s[1]$
	T_r	$T_\alpha + T_p \in \mathfrak{g}^*[2] \oplus \mathbb{R}_p[2]$
	T_α	$T_\alpha \in \mathfrak{g}^*[3]$
Structure const.	h^r_I	$\mu_1 = \mathrm{id} : \mathbb{R}^*[1] \to \mathbb{R}^*$
	g^{Ir}	$\mu_1 = \mathrm{id} : \mathbb{R}[2] \to \mathbb{R}[1]$
	k^α_r	$\mu_1 = \mathrm{id} : \mathfrak{g}^*[3] \to \mathfrak{g}^*[2]$
	f^r_{st}	$\mu_2 : \mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{g} : \ \mu_2(x_1,x_2) := [x_1,x_2]$
	d^I_{rs}	$-\kappa : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}[1] : -\kappa(x_1,x_2) := -(x_1,x_2)$
	b_{Irs}	$-\kappa : \mathfrak{g} \otimes \mathbb{R}^*[1] \to \mathfrak{g}^*[2] : -\kappa(x,s) := -2\langle(-,x),s\rangle$
	$c^t_{\alpha s}$	$-\kappa : \mathfrak{g} \otimes \mathfrak{g}^*[2] \to \mathfrak{g}^*[3] : -\kappa(x,z) := -z([x,-])$

The additional constants k^α_r and $c^t_{\alpha s}$ relate to the additional $\mathfrak{g}^*[3]$ and appear in [56] with the additional condition

$$\tfrac{1}{2}b_{Irs}g^{It} - f_{rs}^{t} = -k_r^{\alpha}c_{\alpha s}^{t} \ . \tag{5.11}$$

It is straightforward to check that this, as well as conditions (5.1), (5.2) and (5.3) are satisfied, so that we indeed have an appropriate gauge structure for a (1,0)-model.

5.3 Kinematical data: Gauge sector

The relevant field content of our action contains the categorified connection valued in the 4-term L_∞-algebra $\widehat{\mathfrak{string}}_{\rm ext}(\mathfrak{g})$ of Section 5.2, whose field strengths are defined similarly to those of [47].

Given any metric Lie algebra $\mathfrak{g}$, the higher connection consists of the fields

$$\begin{aligned} &A \in \Omega^1(\mathbb{R}^{1,5}) \otimes (\mathfrak{g} \oplus \mathbb{R}^*) \ , && B \in \Omega^2(\mathbb{R}^{1,5}) \otimes (\mathbb{R}[1] \oplus \mathbb{R}^*[1]) \ , \\ &C \in \Omega^3(\mathbb{R}^{1,5}) \otimes (\mathfrak{g}^*[2] \oplus \mathbb{R}[2]) \ , && D \in \Omega^4(\mathbb{R}^{1,5}) \otimes \mathfrak{g}^*[3] \ , \end{aligned} \tag{5.12}$$

where D will be a non-propagating 4-form potential. As usual, the difference between form degree and L_∞-algebra degree is always 1, cf. Section 3.2. The corresponding curvatures read as

$$\begin{aligned} \mathcal{F} &= \mathrm{d}A + \tfrac{1}{2}\mu_2(A,A) + \mu_1(B) && \in \ \Omega^2(\mathbb{R}^{1,5}) \otimes (\mathfrak{g} \oplus \mathbb{R}^*) \ , \\ \mathcal{H} &= \mathrm{d}B - \kappa(A,\mathrm{d}A) - \tfrac{1}{3}\kappa(A,\mu_2(A,A)) + \mu_1(C) && \in \ \Omega^3(\mathbb{R}^{1,5}) \otimes (\mathbb{R}[1] \oplus \mathbb{R}^*[1]) \ , \\ \mathcal{G} &= \mathrm{d}C + \mu_2(A,C) + \kappa(\mathcal{F},B) + \mu_1(D) && \in \ \Omega^4(\mathbb{R}^{1,5}) \otimes (\mathfrak{g}^*[2] \oplus \mathbb{R}[2]) \ , \\ \mathcal{I} &= \mathrm{d}D + \kappa(\mathcal{F},C) + \dots && \in \ \Omega^5(\mathbb{R}^{1,5}) \otimes \mathfrak{g}^*[3] \ , \end{aligned} \tag{5.13}$$

where the definitions of all the relevant maps are found in (5.7), (5.8) and (5.10) and the remaining terms in $\mathcal{I}$ are of no importance to our discussion. The difference of degrees of the curvature forms is always 2.

The Bianchi identities in (3.50) take an extended form and are readily computed to be

$$\begin{aligned} \nabla\mathcal{F} - \mu_1(\mathcal{H}) = 0 \ , \qquad & \mathrm{d}\mathcal{H} + \kappa(\mathcal{F},\mathcal{F}) - \mu_1(\mathcal{G}) = 0 \ , \\ \nabla\mathcal{G} - \kappa(\mathcal{F},\mathcal{H}) - \mu_1(\mathcal{I}) = 0 \ , \qquad & \nabla\mathcal{I} + \kappa(\mathcal{F},\mathcal{G}) + \ldots = 0 \ . \end{aligned} \tag{5.14}$$

Here, ∇ is the covariant derivative

$$\nabla\phi \coloneqq \mathrm{d}\phi + \mu_2(A,\phi) \tag{5.15}$$

for any field ϕ taking values in $\widehat{\mathfrak{string}}_{\mathrm{ext}}(\mathfrak{g})$. The combination of $\nabla + \mu_1$ as a differential operator of total degree 1 is very natural in higher gauge theory, cf. e.g. the last section of [109]. The additional terms involving κ are present as we are using the twisted curvatures of Section 3.5 instead of the canonical ones. This also motivates the introduction of the following generalized notions of variation, cf. [47]:

$$\Delta B \coloneqq \delta B - \kappa(\delta A, A) \ , \quad \Delta C \coloneqq \delta C + \kappa(\delta A, B) \ , \quad \Delta D \coloneqq \delta D - \kappa(\delta A, C) \ , \tag{5.16}$$

which allows us to write

$$\begin{aligned} \delta\mathcal{F} &= \nabla\delta A + \mu_1(\Delta B) \\ \delta\mathcal{H} &= \mathrm{d}(\Delta B) - 2\kappa(\mathcal{F},\delta A) + \mu_1(\Delta C) \ , \\ \delta\mathcal{G} &= \nabla(\Delta C) + \kappa(\delta A,\mathcal{H}) + \kappa(\mathcal{F},\Delta B) + \mu_1(\Delta D) \ , \\ \delta\mathcal{I} &= \mathrm{d}\delta D + \kappa(\mathcal{F},\delta C) + \kappa(\nabla\delta A, C) + \ldots \ . \end{aligned} \tag{5.17}$$

Infinitesimal gauge transformations are parameterized by

$$\begin{aligned} \alpha \in \Omega^0(\mathbb{R}^{1,5}) \otimes (\mathfrak{g} \oplus \mathbb{R}^*) \ , \qquad & \Lambda \in \Omega^1(\mathbb{R}^{1,5}) \otimes (\mathbb{R}[1] \oplus \mathbb{R}^*[1]) \ , \\ \Sigma \in \Omega^2(\mathbb{R}^{1,5}) \otimes (\mathfrak{g}^*[2] \oplus \mathbb{R}[2]) \ , \qquad & \Xi \in \Omega^3(\mathbb{R}^{1,5}) \otimes \mathfrak{g}^*[3] \ , \end{aligned} \tag{5.18}$$

and they modify gauge potentials and their curvatures as follows:

$$
\begin{aligned}
\delta A &= \mathrm{d}\alpha + \mu_2(A,\alpha) - \mu_1(\Lambda) \ , & \delta\mathcal{F} &= \mu_2(\mathcal{F},\alpha) \ , \\
\delta B &= \mathrm{d}\Lambda + \kappa(\mathcal{F},\alpha) - \tfrac{1}{2}\mu_3(A,A,\alpha) - \mu_1(\Sigma) \ , & \delta\mathcal{H} &= 0 \ , \\
\delta C &= \mathrm{d}\Sigma + \mu_2(C,\alpha) + \mu_2(A,\Sigma) + \kappa(\mathcal{F},\Lambda) - \mu_1(\Xi) \ , & \delta\mathcal{G} &= \mu_2(\mathcal{G},\alpha) \ , \\
\delta D &= \mathrm{d}\Xi - \kappa(\mathcal{F},\Sigma) + \dots \ , & \delta\mathcal{I} &= \mu_2(\mathcal{I},\alpha) \ .
\end{aligned}
\tag{5.19}
$$

5.4 Kinematical data: Supersymmetry partners

Let us now complete the above field content by adding the remaining fields of the full $\mathcal{N} = (2,0)$ tensor supermultiplet and introduce $\mathcal{N} = (1,0)$ superpartners for the 1-form gauge potential. We use the same fields and spinor conventions as [47, 55], see also [110].

The R-symmetry group for $\mathcal{N} = (1,0)$ supersymmetry is $\mathsf{Sp}(1)$, and therefore all fields arrange in a representation of this group. We use i, j as indices for representations of $\mathsf{Sp}(1)$. R-symmetry indices are raised and lowered using the Levi–Civita symbol ε^{ij}. For a list of all relevant conventions see Appendix B.1.

First, we have the vector supermultiplet containing the one-form gauge potential A, a doublet of symplectic Majorana–Weyl spinors λ^i, satisfying $\gamma_7\lambda^i = \lambda^i$, as well as a triplet of auxiliary scalar fields $Y^{ij} = Y^{(ij)}$, all taking values in $\mathfrak{g} \oplus \mathbb{R}^*$. Supersymmetry transformations are parameterized by a doublet of chiral spinor ε^i with $\gamma_7\varepsilon^i = \varepsilon^i$ and read as

$$
\begin{aligned}
\delta A &= -\bar{\varepsilon}\gamma_{(1)}\lambda \ , \\
\delta\lambda^i &= \tfrac{1}{8}\gamma^{\mu\nu}\mathcal{F}_{\mu\nu}\varepsilon^i - \tfrac{1}{2}Y^{ij}\varepsilon_j + \tfrac{1}{4}\mu_1(\phi)\varepsilon^i, \\
\delta Y^{ij} &= -\bar{\varepsilon}^{(i}\gamma^\mu\nabla_\mu\lambda^{j)} + 2\mu_1(\bar{\varepsilon}^{(i}\chi^{j)}) \ ,
\end{aligned}
\tag{5.20}
$$

where we used the notation $\gamma_{(p)} = \mathrm{d}x^{\mu_1} \wedge \dots \wedge \mathrm{d}x^{\mu_p}\gamma_{\mu_1}\dots\gamma_{\mu_p}$. We also suppressed all evident R-symmetry index contractions.

Infinitesimal gauge transformations, parameterized by $(\alpha, \Lambda, \Sigma, \Xi)$, act on the additional fields in the vector multiplet according to

$$\delta\lambda = \mu_2(\lambda, \alpha) \qquad \text{and} \qquad \delta Y^{ij} = \mu_2(Y^{ij}, \alpha)\ . \tag{5.21}$$

Second, we have the tensor supermultiplet containing the two-form gauge potential B, a doublet of Majorana–Weyl spinors χ^i satisfying $\gamma_7\chi^i = -\chi^i$ and a single scalar field ϕ, all taking values in $\mathbb{R}[1] \oplus \mathbb{R}^*[1]$. The supersymmetry transformations read as

$$\begin{aligned}
\delta\phi &= \bar{\varepsilon}\chi\ ,\\
\delta\chi^i &= \tfrac{1}{48}\gamma^{\mu\nu\rho}\mathcal{H}_{\mu\nu\rho}\varepsilon^i + \tfrac{1}{4}\not{d}\phi\varepsilon^i + \tfrac{1}{2}\kappa(\gamma^\mu\lambda^i, \bar{\varepsilon}\gamma_\mu\lambda)\ ,\\
\Delta B &= -\bar{\varepsilon}\gamma_{(2)}\chi\ .
\end{aligned} \tag{5.22}$$

Note that gauge transformations act trivially on the fields χ and ϕ.

We also have the following supersymmetry transformation for the 3-form potential C:

$$\Delta C = \kappa(\bar{\varepsilon}\gamma_{(3)}\lambda, \phi)\ . \tag{5.23}$$

So far we have introduced the (1,0) vector and (1,0) tensor multiplet forming the full field content of the (1,0) model in [47]. In order to arrive at the field content of the (2,0) theory we need to extend this by a (1,0) hypermultiplet, which contains the matter fields. A detailed review of the general situation is found in [55]. In the following, we only repeat what is necessary for our construction which has a flat target space.

We start by embedding our gauge Lie algebra $\mathfrak{g}$ into $\mathfrak{sp}(n)$. Recall that the group $\mathsf{Sp}(n) \cong \mathsf{USp}(2n)$ is given by $2n \times 2n$-dimensional unitary complex matrices m such $m^T\Omega m = \Omega$ and we choose

$$\Omega = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix} \tag{5.24}$$

for the symplectic form. Therefore the Lie algebra $\mathfrak{sp}(n) = \mathfrak{usp}(2n)$ consists of complex block matrices

$$u = \begin{pmatrix} A & B \\ -B^* & -A^T \end{pmatrix} \qquad \text{with} \qquad A^\dagger = -A \qquad \text{and} \qquad B^T = B \tag{5.25}$$

and as a vector space, it has dimension $n^2+n(n+1) = n(2n+1)$. By putting $B = 0$ in (5.25), we obtain an embedding $\mathfrak{u}(n)\hookrightarrow\mathfrak{sp}(n)$. Demanding that A is a real matrix leads to an embedding $\mathfrak{so}(n)\hookrightarrow\mathfrak{sp}(n)$ and considering subgroups of $\mathfrak{u}(n)$ leads to the E-series. Altogether, we can indeed embed any of the Lie algebras of types ADE into $\mathfrak{sp}(n)$, and we denote the generators of the original Lie algebra $\mathfrak{g}$ in this matrix embedding by $u_\alpha{}^a{}_b$, where again $\alpha = 1,\ldots,\dim(\mathfrak{g})$ and $a,b = 1,\ldots,2n$.

We are particularly interested in the cases $\mathfrak{g} = \mathfrak{su}(N)$ and $\mathfrak{g} = \mathfrak{u}(N)\times\mathfrak{u}(N)$ and we shall embed both cases into $\mathfrak{u}(N^2)\subset\mathfrak{sp}(N^2)$ to obtain adjoint and bifundamental representations[1]. Correspondingly, we have $\mathbb{R}^{2\times 2N^2}$ scalar fields encoded in $2\times 2n$-dimensional matrices q^{ia}, $i = 1,2$, $a = 1,\ldots,2n$, where the index i labels a vector of the R-symmetry group $\mathsf{Sp}(1)$. The superpartners of the scalar fields are $2n$ antichiral, symplectic Majorana spinors ψ^a, satisfying $\gamma_7\psi = -\psi$.

This choice of the number of hypermultiplets arises from doubling the degrees of freedom in the $\mathfrak{su}(N)$-valued $(1,0)$-vector multiplet and the $\mathfrak{u}(1)$-valued $(1,0)$-tensor multiplet that we have in our theory. More justification arises from the dimensional reductions discussed in Sections 5.8 and 5.9.

We now define the obvious action of the gauge Lie algebra $\mathfrak{g}$ on $\mathbb{R}^{2n}$ by

$$\rhd\colon \mathfrak{g}\times\mathbb{R}^{2n}\to\mathbb{R}^{2n}~,\qquad ((x^\alpha\tau_\alpha)\rhd w)^a = x^\alpha u_\alpha{}^a{}_b w^b~, \tag{5.27}$$

where τ_α are the generators of $\mathfrak{g}$. We shall also use the bilinear pairing

$$\prec -,-\succ\colon \mathbb{R}^{2n}\times\mathbb{R}^{2n}\to\mathbb{R}~,\qquad \prec w_1,w_2\succ := \Omega_{ab}w_1^a w_2^b \tag{5.28}$$

for all $w_1,w_2\in\mathbb{R}^{2n}$. Infinitesimal gauge transformations, parameterized again by $(\alpha,\Lambda,\Sigma,\Xi)$, act then on the fields in the hypermultiplets as

$$\delta q^i = \alpha\rhd q^i \qquad\text{and}\qquad \delta\psi = \alpha\rhd\psi~, \tag{5.29}$$

[1]Recall that this can be done via vectorization and the Kronecker product: given matrices A,B,C,D, we have

$$ABC = D \quad\Leftrightarrow (C^T\otimes A)\mathrm{vec}(B) = \mathrm{vec}(D)~, \tag{5.26}$$

where $\mathrm{vec}(B)$ is the vector consisting of the columns of B stacked on top of each other. In particular, if λ_α is a generator of $\mathfrak{su}(N)$ in the fundamental representation, then $\mathbb{1}\otimes\lambda_\alpha - \lambda_\alpha^T\otimes\mathbb{1}$ is the corresponding generator in the vectorization of the adjoint.

and the covariant derivatives are given by

$$\nabla q^i := \mathrm{d}q^i + A \rhd q^i \qquad \text{and} \qquad \nabla\psi := \mathrm{d}\psi + A \rhd \psi \ . \tag{5.30}$$

The supersymmetry transformations for the fields in the hypermultiplets read as

$$\delta q^{ia} = \bar{\varepsilon}^i \psi^a \qquad \text{and} \qquad \delta\psi^a \ = \tfrac{1}{2}\nabla\!\!\!/ q^{ia}\varepsilon_i \ . \tag{5.31}$$

Finally, we introduce duality invariant supersymmetric extensions of the curvature forms (5.13), which will become convenient later:

$$\begin{aligned} \mathscr{H} &:= *\mathcal{H} - \mathcal{H} - \kappa(\bar{\lambda}, \gamma_{(3)}\lambda) \ , \\ \mathscr{G} &:= \mathcal{G} - \kappa(*\mathcal{F}, \phi) + 2\kappa(\bar{\lambda}, *\gamma_{(2)}\chi) \ , \\ \mathscr{I} &:= \mathcal{I} + \mu_2(\kappa(\bar{\lambda}, \phi), \gamma^\mu\lambda)\mathrm{vol}_\mu + 2 \prec - \rhd q, *\nabla q \succ \\ &\quad\ - 2 \prec \bar{\psi}, - \rhd \gamma^\mu \psi \succ \mathrm{vol}_\mu \ , \end{aligned} \tag{5.32}$$

where $\mathrm{vol}_\mu = \iota_{\frac{\partial}{\partial x^\mu}}\mathrm{vol}$ is the contraction of the volume form on $\mathbb{R}^{1,5}$ with $\frac{\partial}{\partial x^\mu}$, terms like $\prec - \rhd q, *\nabla q \succ$ denote elements of $\mathfrak{g}^*[3]$ and $*$ now denotes the Hodge dual in six dimensions[2]. We choose a convention such that $\gamma_{(3)}$ is anti-self-dual.

5.5 Dynamics: Action

Our action is composed of ingredients collected from [47, 55, 56] and consists of four parts:

$$S = \int_{\mathbb{R}^{1,5}} \mathcal{L}_{\mathrm{tensor}} \ + \ \mathcal{L}_{\mathrm{top}} \ + \ \mathcal{L}_{\mathrm{hyper}} \ + \ \mathcal{L}_{\mathrm{PST}} \ . \tag{5.33}$$

We shall now explain these terms in detail, using the notation, maps and fields defined in sections 5.2, 5.3 and 5.4.

The first part, $\mathcal{L}_{\mathrm{tensor}}$, consists of the terms coupling the $(1,0)$-tensor multiplet

[2] In this chapter, as we are now working in six dimensions, $*$ will always denote the Hodge dual in six dimensions unless specified otherwise.

to the $(1,0)$-vector multiplet and reads as

$$\begin{aligned}\mathcal{L}_{\text{tensor}} = &-\langle \mathrm{d}\phi, *\mathrm{d}\phi\rangle - 4\mathrm{vol}\langle\bar{\chi}, \not{\mathrm{d}}\chi\rangle - \tfrac{1}{2}\langle\mathcal{H}, *\mathcal{H}\rangle + \langle\mathcal{H}, \kappa(\bar{\lambda}, *\gamma_{(3)}\lambda)\rangle \\ &- 2\big\langle\phi\,,\, \kappa(\mathcal{F}, *\mathcal{F}) - 2\mathrm{vol}\,\kappa(Y_{ij}, Y^{ij}) + 4\mathrm{vol}\,\kappa(\bar{\lambda}, \not{\nabla}\lambda)\big\rangle \\ &+ 8\big\langle\kappa(\bar{\lambda}, \mathcal{F}), *\gamma_{(2)}\chi\big\rangle - 16\mathrm{vol}\big\langle\kappa(Y_{ij}, \bar{\lambda}^i), \chi^j\big\rangle\ .\end{aligned} \tag{5.34}$$

Most of this action is expected given our gauge structure. The term $\langle\phi, \kappa(\mathcal{F}, *\mathcal{F})\rangle$, e.g., was essentially conjectured already in [23], and $\mathcal{L}_{\text{tensor}}$ contains its supersymmetric completion.

The second part, $\mathcal{L}_{\text{top}}$, is a complementing topological term,

$$\mathcal{L}_{\text{top}} = \langle\mu_1(C), \mathcal{H}\rangle + \big\langle B, \kappa(\mathcal{F}, \mathcal{F})\big\rangle\ . \tag{5.35}$$

This topological term is due to the presence of the additional, Chern–Simons-like terms in the curvatures (5.13). It can also be seen as arising from the boundary contribution of a manifestly gauge invariant 7-form given by

$$\mathrm{d}\mathcal{L}_{\text{top}} = \langle\mu_1(\mathcal{G}), \mathcal{H}\rangle + \langle\mathcal{H}, \kappa(\mathcal{F}, \mathcal{F})\rangle\ . \tag{5.36}$$

The third part, $\mathcal{L}_{\text{hyper}}$, contains the kinetic and coupling terms for the $(1,0)$-hyper multiplet:

$$\begin{aligned}\mathcal{L}_{\text{hyper}} = &- \prec \nabla q, *\nabla q \succ + 2\mathrm{vol} \prec \bar{\psi}, \not{\nabla}\psi \succ + 8\mathrm{vol} \prec \bar{\psi}, \lambda_i \rhd q^i \succ \\ &+ 2\mathrm{vol} \prec q^i, Y_{ij} \rhd q^j \succ\ .\end{aligned} \tag{5.37}$$

This part of the Lagrangian can be multiplied with any factor without breaking the supersymmetry. Here, we normalize such that the kinetic terms of q and ϕ have the same coefficient, as would be the case if the $\mathsf{Spin}(5) = \mathsf{Sp}(2)$ R-symmetry of the (2,0)-theory was realized.

Finally, the PST mechanism which lets the self-duality of $\mathcal{H}$ appear as an equation of motion is implemented by adding the last part, $\mathcal{L}_{\text{PST}}$. In order to be explicit, let us introduce the pairing

$$\Phi : \mathfrak{g}^*[2] \oplus \mathbb{R}_p[2] \to \mathfrak{g} \oplus \mathbb{R}_q^*, \quad \Phi(y+p) := \frac{1}{\phi_s}(y, -)_* \ , \tag{5.38}$$

where $(y,-)_* : \mathfrak{g}^* \to \mathfrak{g}$ is the inverse of the Killing form $(x,-) : \mathfrak{g} \to \mathfrak{g}^*$ and $\phi_s := \phi|_{\mathbb{R}_s^*[1]}$. Clearly, Φ is only defined if $\phi_s \neq 0$. At this point the PST mechanism leading to generalized duality equations is no longer fully defined. This is a symptom of the tensionless string phase transition[111, 112], see also [55].

Given Φ, the PST term of the action reads as

$$\mathcal{L}_{\mathrm{PST}} = \tfrac{1}{2}\big\langle \iota_V \mathscr{H}, \mathscr{H} \big\rangle \wedge v + \langle \Phi(\iota_V * \mathscr{G}), * \, \iota_V * \mathscr{G} \rangle \ , \tag{5.39}$$

where the duality invariant supersymmetrically extended higher curvatures $\mathscr{H}$ and $\mathscr{G}$ were defined in (5.32). Additionally, v is a nowhere vanishing exact auxiliary one-form and V its corresponding dual vector field:

$$v = v_\mu \mathrm{d}x^\mu = \mathrm{d}a, \quad \iota_V v = 1 \ , \quad \iota_V * v = 0 \tag{5.40}$$

for some auxiliary scalar field[3] a. These additional terms allow for a manifestly Lorentz-invariant Lagrangian that includes the expected duality equations in its equations of motion without having to impose these by hand, see [53, 54] for original references and, furthermore, [114–117] for follow-ups.

5.6 Dynamics: Equations of motion

In the following we give an outline of the computation implementing the PST mechanism in our model. This is an extension of the purely bosonic computations of [56] to the supersymmetric case, which is possible due to the simplifications introduced by our use of the 4-term L_∞-algebra $\widehat{\mathfrak{string}}_{\mathrm{ext}}(\mathfrak{g})$. For more details on the calculations and the full Lagrangian variation see Appendix B.2. Additionally, there are a number of useful identities repeatedly used in this calculation, that we list in Appendix B.1.

[3] In general, the auxiliary field v is only required to be closed, see e.g. [53, 113]. As we are working on $\mathbb{R}^{1,5}$ we take it to be exact.

We start with the relevant part of the action, $\mathcal{L}_{\text{PST}}$, which is complemented by the terms including $\mathcal{H}$ from $\mathcal{L}_{\text{tensor}}$. We can combine both into

$$\begin{aligned}\mathcal{L}'_{\text{PST}} &= \mathcal{L}_{\text{PST}} - \tfrac{1}{2}\langle\mathcal{H}, *\mathcal{H}\rangle + \langle\mathcal{H}, \kappa(\bar{\lambda}, *\gamma_{(3)}\lambda)\rangle \\ &= -\big\langle\iota_V(*\mathcal{H} - \mathcal{H} - 2\kappa(\bar{\lambda}, \gamma_{(3)}\lambda)), \mathcal{H}\big\rangle \wedge v + \big\langle\Phi(\iota_V * \mathscr{G}), *(\iota_V * \mathscr{G})\big\rangle\ ,\end{aligned} \tag{5.41}$$

where we also use the fact that $\gamma_{(3)}$ is anti-self-dual. The variation of this expression is readily computed to be

$$\begin{aligned}\delta\mathcal{L}'_{\text{PST}} = -2\big\langle\iota_V\mathscr{H}, \delta\mathcal{H} - \tfrac{1}{2}\delta v \wedge \iota_V\mathscr{H}\big\rangle \wedge v + \langle\mathcal{H}, \delta\mathcal{H}\rangle \\ - 2\langle\Phi(\iota_V * \mathscr{G}), \iota_V\mathscr{G}\rangle \wedge v \wedge \delta v + 2\langle\Phi(\iota_V * \mathscr{G}), \delta\mathcal{G}\rangle \wedge v \\ - 2\langle\Phi(\mathscr{G}), \kappa(\delta\mathcal{F}, \phi)\rangle - 2\langle\Phi(\iota_V\mathscr{G}), \kappa(\delta\mathcal{F}, \phi)\rangle \wedge v \\ + \delta_{\phi,\lambda,\chi}\mathcal{L}'_{\text{PST}}\ ,\end{aligned} \tag{5.42}$$

where we take into account that $\iota_V\delta v = 0$ as $\iota_V v = 1$. Inserting the variations of the curvatures given in (5.17) and using the Bianchi identities (5.14) to simplify expressions leads to

$$\begin{aligned}\delta\mathcal{L}'_{\text{PST}} = 2\big\langle\Phi(\iota_V * \mathscr{G} \wedge v), \mu_1(\Delta D)\big\rangle + \big\langle\mu_1(2\iota_V\mathscr{H} \wedge v + \mathcal{H}) - 2\nabla\Phi(\iota_V * \mathscr{G}) \wedge v, \Delta C\big\rangle \\ + \big\langle 2\mathrm{d}(\iota_V\mathscr{H} \wedge v) - \kappa(\mathcal{F}, \mathcal{F}) + \mu_1(\mathscr{G}) + 4\kappa(\Phi(\iota_V * \mathscr{G} \wedge v), \mathcal{F}), \Delta B\big\rangle \\ - \big\langle\kappa(\mathcal{F}, 2\iota_V\mathscr{H} \wedge v + \mathcal{H}) + 2\nabla(\kappa(\Phi(\mathscr{G}), \phi) + \kappa(\Phi(\iota_V\mathscr{G} \wedge v), \phi)) \\ + 2\kappa(\Phi(\iota_V * \mathscr{G} \wedge v), \mathcal{H}), \delta A\big\rangle - (\big\langle\iota_V\mathscr{H}, \iota_V\mathscr{H}\big\rangle \\ + 2\big\langle\Phi(\iota_V * \mathscr{G}), \iota_V\mathscr{G}\big\rangle) \wedge v \wedge \delta v + \delta_{\phi,\lambda,\chi}\mathcal{L}'_{\text{PST}}\ .\end{aligned} \tag{5.43}$$

Note that we can cancel some of the terms originating from $\langle\mathcal{H}, \delta\mathcal{H}\rangle$ by adding the variation of the topological term, which is given by

$$\delta\mathcal{L}_{\text{top}} = \big\langle\kappa(\mathcal{F}, \mathcal{F}) + \mu_1(\mathcal{G}), \Delta B\big\rangle - \big\langle\mu_1(\mathcal{H}), \Delta C\big\rangle - \big\langle\kappa(\mathcal{F}, \mathcal{H}), \delta A\big\rangle\ . \tag{5.44}$$

Furthermore, there are additional terms for the variation with respect to the gauge potential A coming from both $\mathcal{L}_{\text{tensor}}$ and $\mathcal{L}_{\text{hyper}}$. After including these terms, and again using the Bianchi identities (5.14) to simplify expressions, we arrive at

$$\begin{aligned}\delta\mathcal{L} = {} & 2\big\langle\Phi(\iota_V * \mathscr{G} \wedge v), \mu_1(\Delta D)\big\rangle + \big\langle 2\mu_1(\iota_V\mathscr{H} \wedge v) - 2\nabla\Phi(\iota_V * \mathscr{G}) \wedge v, \Delta C\big\rangle \\ & + \big\langle 2\mathrm{d}(\iota_V\mathscr{H} \wedge v) + 2\mu_1(\mathscr{G}) + 4\kappa(\Phi(\iota_V * \mathscr{G} \wedge v), \mathcal{F}), \Delta B\big\rangle \\ & + \big\langle \mu_1(\mathscr{I}) - 2\kappa(\mathcal{F}, \iota_V\mathscr{H} \wedge v) + 2\nabla(\kappa(\Phi(\iota_V\mathscr{G} \wedge v), \phi)) \\ & - 2\kappa(\Phi(\iota_V * \mathscr{G} \wedge v), \mathcal{H}), \delta A\big\rangle - \big(\big\langle\iota_V\mathscr{H}, \iota_V\mathscr{H}\big\rangle + 2\big\langle\Phi(\iota_V * \mathscr{G}), \iota_V\mathscr{G}\big\rangle\big) \wedge v \wedge \delta v \\ & + \delta_{\phi,\chi,\lambda,Y,q,\psi}\mathcal{L} \ ,\end{aligned} \tag{5.45}$$

where we also used $\nabla(\mathscr{G} + \kappa(\Phi(\mathscr{G}), \phi)) = \mathrm{d}^2 C = 0$. The remaining terms can be found in equation (B.13). Given this it is immediate that the Lagrangian is invariant under any one of the symmetry transformations

$$\delta A = \varphi_A \wedge v \ , \quad \Delta B = \varphi_B \wedge v \ , \quad \Delta C = \varphi_C \wedge v \ , \quad \Delta D = \varphi_D \wedge v \ , \tag{5.46}$$

where φ_C and φ_D are free parameters, while φ_A is restricted to lie in $\mathbb{R}_q^*$ and φ_B is restricted to lie in $\mathbb{R}_r[1]$. Furthermore, it can be shown using the above variation that the Lagrangian is invariant under the combined transformations

$$\begin{gathered}\delta v = \mathrm{d}\varphi_v(x) \ , \quad \delta A = \varphi_v(x)\Phi(\iota_V * \mathscr{G}) \ , \\ \Delta B = \varphi_v(x)\iota_V\mathscr{H} \ , \quad \Delta C = -\varphi_v(x)\iota_V\mathscr{G} \ , \\ \Delta D = \tfrac{1}{2}\varphi_v(x)\iota_V\mathscr{I} \ ,\end{gathered} \tag{5.47}$$

where φ_v is a function on $\mathbb{R}^{1,5}$. This symmetry transformation exposes the auxiliary nature of v, guaranteeing that no additional degrees of freedom are introduced.

Let us now come to the derivation of the duality equations from the variation (5.45). Starting with the variation with respect to $\mu_1(\Delta D)$ we have

$$\Phi(\iota_V * \mathscr{G}) \wedge v\big|_{\mathfrak{g}} = 0 \ . \tag{5.48}$$

Since, by construction, $\iota_V * \mathscr{G}$ has no common directions with v and, furthermore, the kernel of Φ lies in $\mathbb{R}[2]$, this is equivalent to

$$\iota_V * \mathscr{G}\big|_{\mathfrak{g}^*[2]} = 0 \ . \tag{5.49}$$

Additionally, we can use the last symmetry in (5.46) to gauge away $\iota_V \mathscr{G}\big|_{\mathfrak{g}^*[2]}$. Indeed, from (5.17) we have,

$$\iota_V \delta_D \mathscr{G} \wedge v = \iota_V \mu_1(\Delta D) \wedge v = -\mu_1(\varphi_D) \wedge v \ . \tag{5.50}$$

Thus, choosing $\mu_1(\varphi_D) = -\iota_V \mathscr{G}$ we gauge-fix $\iota_V \mathscr{G}\big|_{\mathfrak{g}^*[2]} = 0$, which in conjunction with equation (5.49) implies

$$\mathscr{G}\big|_{\mathfrak{g}^*[2]} = 0 \ . \tag{5.51}$$

This reduces the variation with respect to ΔC to the equation

$$\mu_1(\iota_V \mathscr{H} \wedge v) = 0 \ . \tag{5.52}$$

As, again by construction, $\iota_V \mathscr{H}$ does not share directions with v, we can write this as

$$\iota_V \mathscr{H}\big|_{\mathbb{R}^*_s[1]} = 0 \ . \tag{5.53}$$

Taking (5.51) into account and turning our attention to the variation with respect to ΔB we have

$$\mathrm{d}(\iota_V \mathscr{H} \wedge v) + \mu_1(\mathscr{G}) = 0 \ . \tag{5.54}$$

This immediately implies $\mu_1(\mathscr{G}) \wedge v = 0$ which is equivalent to

$$\iota_V * \mathscr{G}\big|_{\mathbb{R}_p[2]} = 0 \ . \tag{5.55}$$

Additionally, using the third symmetry in (5.46) we have

$$\iota_V \delta_C \mathscr{G} \wedge v\big|_{\mathbb{R}_p[2]} = \iota_V \mathrm{d}\Delta C \wedge v\big|_{\mathbb{R}_p[2]} = -\mathrm{d}\varphi_C \wedge v\big|_{\mathbb{R}_p[2]} \ , \tag{5.56}$$

which when choosing $\varphi_C = -\iota_V \mathscr{G}\big|_{\mathbb{R}_p[2]}$ allows one to gauge-fix to $\iota_V \mathscr{G}\big|_{\mathbb{R}_p[2]} = 0$. This, together with (5.51) and (5.55), leads to the first duality equation $\mathscr{G} = 0$.

Using this leaves the variation with respect to ΔB with the equation

$$\mathrm{d}(\iota_V \mathscr{H} \wedge v)\big|_{\mathbb{R}_r[1]} = 0 \ , \tag{5.57}$$

which has the general solution

$$\iota_V \mathscr{H} \wedge v\big|_{\mathbb{R}_r[1]} = \mathrm{d}\tilde{\varphi} \wedge v \ , \tag{5.58}$$

where $\tilde{\varphi} \in \Omega^1(\mathbb{R}^{1,5}) \otimes \mathbb{R}_r[1]$. Note that using (5.17) we also have under the first symmetry in (5.46) that

$$\iota_V \delta\mathscr{H} = \iota_V(*(\mathrm{d}\varphi_B \wedge v) - \mathrm{d}\varphi_B \wedge v) \wedge v = -\mathrm{d}\varphi_B \wedge v \ , \tag{5.59}$$

where φ_B is also an element of $\Omega^1(\mathbb{R}^{1,5}) \otimes \mathbb{R}_r[1]$. As this has the same form as the general solution above, we can gauge-fix to $\varphi_B = -\tilde{\varphi}$ and arrive at the self-duality equation $\mathscr{H} = 0$. Lastly, looking at the variation with respect to δA and taking into account all equations of motion we have derived so far, we are left with the last duality equation $\mathscr{I} = 0$. Note that this leaves the equations coming from the variation with respect to δv trivially satisfied.

The remaining equations of motion are straightforward to calculate, see Appendix B.2. Altogether, we arrive at the set of equations

$$\begin{aligned}
0 &= \mathscr{H} = *\mathcal{H} - \mathcal{H} - \kappa(\bar{\lambda}, *\gamma_{(3)}\lambda) \ , \\
0 &= \mathscr{G} = \mathcal{G} - \kappa(*\mathcal{F}, \phi) + 2\kappa(\bar{\lambda}, *\gamma_{(2)}\chi) \ , \\
0 &= \mathscr{I} = \mathcal{I} + \mu_2(\kappa(\bar{\lambda}, \phi), \gamma^\mu\lambda)\mathrm{vol}_\mu + 2 \prec - \rhd q, *\nabla q \succ \\
&\qquad - 2 \prec \bar{\psi}, - \rhd \gamma^\mu \psi \succ \mathrm{vol}_\mu \ ,
\end{aligned} \tag{5.60}$$

together with the remaining equations of motion for the tensor multiplet,

$$
\begin{aligned}
\kappa(\nabla\!\!\!/\lambda_i,\phi)+\tfrac{1}{2}\kappa(\lambda_i,d\!\!\!/\phi) &= -\tfrac{1}{2}*\kappa(\mathcal{F},*\gamma_{(2)}\chi_i)-\kappa(Y_{ij},\chi^j)+\tfrac{1}{8}*\kappa(*\gamma_{(3)}\lambda_i,\mathcal{H}) \\
&\quad + \mu_1(\prec - \rhd q_i,\psi\succ)\ , \\
\kappa(Y^{ij},\phi)-2\kappa(\bar\lambda^{(i},\chi^{j)}) &= -\tfrac{1}{2}\mu_1(\prec q^{(i},-\rhd q^{j)}\succ)\ , \\
d\!\!\!/\chi_i &= -\tfrac{1}{2}*\kappa(\mathcal{F},*\gamma_{(2)}\lambda_i)+2\kappa(Y_{ji},\lambda^j)\ , \\
\Box\phi &= -*\kappa(\mathcal{F},*\mathcal{F})-\kappa(Y_{ij},Y^{ij})+\kappa(\bar\lambda,\nabla\!\!\!/\lambda)\ ,
\end{aligned}
\tag{5.61}
$$

as well as the equations for the hypermultiplet,

$$
\begin{aligned}
\Box q_i &= -4\bar\lambda_i\rhd\psi-2Y_{ij}\rhd q^j\ , \\
\nabla\!\!\!/\psi &= 2\lambda\rhd q\ .
\end{aligned}
\tag{5.62}
$$

Note that these equations of motion become partially degenerate for $\phi_s=0$; in particular, the duality equation linking $\mathcal{G}$ and $\mathcal{F}$ breaks down. This is again a reflection of the tensionless string phase transition.

5.7 BPS states and formulation for loop model

Ideally, the non-abelian self-dual strings discussed in Chapter 4 should be BPS states of our model. Let us start by setting the fermions and fields in the hypermultiplets to zero and taking all remaining fields to be independent of two directions, say x^0 and x^5. We then have a resulting scalar field $\breve\phi$ and curvatures $\breve{\mathcal{F}}$ and $\breve{\mathcal{H}}$ on the remaining $\mathbb{R}^4$ in the x^1,x^2,x^3 and x^4-directions. In [102], the BPS solutions of the (1,0)-model were calculated: imposing the Killing spinor equations, that is, imposing that the supersymmetry variations of the fermions in (5.20) and (5.22) vanish, reduces the equations of motion to

$$
\begin{aligned}
*_4\mathrm{d}\breve\phi &= \breve{\mathcal{H}}\ , \\
\Box\breve\phi &= -*_4\kappa(\breve{\mathcal{F}},*_4\breve{\mathcal{F}})\ ,
\end{aligned}
\tag{5.63}
$$

where the second line originates from the last equation in (5.61). Comparing with the discussion in Section 4.5, it is then clear that solutions to the equations (4.38) and (4.40) are indeed BPS states in our model.

As discussed in Chapter 4, these non-abelian self-dual strings can be equivalently described in the twisted loop model of the string algebra. In the above we have only used the skeletal model $\widehat{\mathfrak{string}}_{\mathrm{sk}}(\mathfrak{g})$ as the basis of our gauge structure as this readily translates to the $(1,0)$-gauge structures of [47]. One would expect that it should be equivalently possible to use the loop model $\widehat{\mathfrak{string}}_{\hat{\Omega}}(\mathfrak{g})$ as the basis of our gauge structure. More specifically, we can minimally extend $\widehat{\mathfrak{string}}_{\hat{\Omega}}(\mathfrak{g})$ to include a cyclic structure and trivially add a copy of $\mathfrak{g}^*$ to allow for a non-propagating potential D. Altogether, one arrives at the complex

$$\begin{array}{ccccccc}
\mathbb{R}^* & \xleftarrow{(0,\mathrm{id})^*} & \widehat{\Omega\mathfrak{g}}^*[1] & \xleftarrow{(\mathrm{id},0)^*} & P_0\mathfrak{g}^*[2] & \xleftarrow{\partial^*} & \mathfrak{g}^*[3] \\
\oplus & & \oplus & & \oplus & & \\
P_0\mathfrak{g} & \xleftarrow{(\mathrm{id},0)} & \widehat{\Omega\mathfrak{g}}[1] & \xleftarrow{(0,\mathrm{id})} & \mathbb{R}[2] & & ,
\end{array} \tag{5.64}$$

together with the appropriately dualized maps.

However, the curvatures for the twisted loop model as above cannot be realized as a $(1,0)$-gauge structure of [47]. More specifically, the constant d^I_{rs} cannot be allowed to be purely symmetric any longer. This indicates that the $(1,0)$-gauge structure does not encapsulate the full picture. One should rather perform the twist as in Section 3.5 for the whole algebra in (5.64) to derive the appropriate expressions for the curvatures, which can serve as a basis for a reformulation of the model. A full such formulation in this picture should therefore shed more light on the situation and would be of interest in future work.

5.8 Reduction to super Yang–Mills theory

Let us now come to the dimensional reduction of our model to a supersymmetric Yang–Mills theory. While, in general, it is not clear how non-abelian Yang–Mills theory can arise from the $\mathbb{R}$-valued curvature $\mathcal{H}$, our model offers a naive and straightforward reduction due to the presence of the 1-form potential A in the vector multiplet.

We start from the L_∞-algebra $\widehat{\mathfrak{string}}_{\rm ext}(\mathfrak{su}(n))$. We then compactify $\mathbb{R}^{1,5}$ along two spatial directions on a torus T^2 with radii R_9 and R_{10} and modular parameter

$$\tau = \frac{\theta}{2\pi} + \frac{\mathrm{i}}{g_{\rm YM}^2} = \tau_1 + \mathrm{i}\tau_2 \ . \tag{5.65}$$

We assume that, analogously to the case of M2-brane models [118], the component $\phi_s \in \mathbb{R}^*[1]$ of ϕ develops a vacuum expectation

$$\langle \phi_s \rangle = -\frac{1}{32\pi^2}\frac{1}{R_{10}^2} = -\frac{1}{32\pi^2}\frac{\tau_2}{R_9 R_{10}} \ , \tag{5.66}$$

which matches the inverse length dimension 2 of the scalar field ϕ. In order to get the full θ-term, we also put a constant 2-form field on the torus:

$$\langle B_s \rangle = \frac{1}{16\pi^2}\frac{\tau_1}{R_9 R_{10}} \mathrm{vol}(T^2) \ . \tag{5.67}$$

We are then interested in the double scaling limit of small radii R_9 and R_{10} with the ratio R_9/R_{10} constant. Note that

$$-2\int_{T^2} \mathrm{vol}\ (T^2)\langle \phi_s \rangle = \frac{1}{4g_s} = \frac{1}{4g_{\rm YM}^2} \qquad \text{and} \qquad \int_{T^2} \langle B_s \rangle = \frac{\tau_1}{4} = \frac{\theta}{8\pi} \ . \tag{5.68}$$

according to the usual relation between compactification radii, string coupling constant and Yang–Mills coupling in four dimensions.

We can regard ϕ_s as the radial coordinate on a cone over the target space $\mathbb{R}^{2\times 2n}$, and scaling ϕ_s involves a dilation of the hyper Kähler cone $\mathbb{R}^{2\times 2n}$. Considering the underlying geometric structures as presented e.g. in [55], we note that the homothetic Killing spinor rescales the metric on $\mathbb{R}^{2\times 2n}$, which is readily identified with a rescaling of the symplectic form Ω defining the bilinear pairing $\prec -, - \succ$, again by a factor of $\frac{1}{\pi^2 R_9 R_{10}}$. In the small radius limit, the dominant terms in the Lagrangian are therefore

$$\mathcal{L}_{R\to 0} = \frac{1}{\pi^2 R_9 R_{10}}\Big[\frac{\tau_1}{4}(F,F) + \frac{\tau_2}{4}\Big((F,*F) - 2\mathrm{vol}(Y_{ij},Y^{ij}) + 4\mathrm{vol}(\bar\lambda,\nabla\!\!\!/\,\lambda) - \prec \nabla q, *\nabla q \succ + 2\mathrm{vol} \prec \bar\psi, \nabla\!\!\!/\,\psi \succ + 8\mathrm{vol} \prec \bar\psi, \lambda_i \rhd q^i \succ + 2\mathrm{vol} \prec q^i, Y_{ij} \rhd q^j \succ \Big)\Big] , \tag{5.69}$$

where we used the fact that $\kappa(\mathcal{F},*\mathcal{F})|_{\mathbb{R}_r[1]} = (F,*F)$ and $\kappa(\mathcal{F},\mathcal{F})|_{\mathbb{R}_r[1]} = (F,F)$ in the case at hand. We can now reduce the six-dimensional gauge potential A to a four dimensional one, $\check{A}$, with curvature $\check{F}$ together with two scalar fields $\check{\sigma}$, which are the components of A along the torus. We also rotate the field content in the hypermultiplet to obtain scalar fields q^i and spinors ψ taking values in the adjoint representation of $\mathfrak{su}(n)$. Finally we integrate out the auxiliary field Y and integrate over the torus to implement the compactification. This yields the Lagrangian

$$\mathcal{L}_{4d} = \frac{1}{4g^2_{\mathrm{YM}}}\Big((\check{F}, *_4\check{F}) + \mathrm{tr}\,(\check{\nabla}\check{\sigma}, *_4\check{\nabla}\check{\sigma}) + 4\mathrm{vol}(\bar\lambda, \check{\nabla\!\!\!/}\lambda) + \mathrm{tr}\,(\check{\nabla}q, *_4\check{\nabla}q) + 4\mathrm{vol}\,\mathrm{tr}\,(\bar\psi, \check{\nabla\!\!\!/}\psi) + 4\mathrm{vol}\,\mathrm{tr}\,(\bar\lambda, [\check{\sigma\!\!\!/}, \lambda]) + 4\mathrm{vol}\,\mathrm{tr}\,(\bar\psi, [\check{\sigma\!\!\!/}, \psi]) + 8\mathrm{vol}\,\mathrm{tr}\,(\bar\psi, [\lambda, q]) + \mathrm{tr}\,([\check{\sigma}_1, \check{\sigma}_2]^2) + \mathrm{tr}\,([\check{\sigma}, q]^2) + \mathrm{tr}\,([q^1, q^2]^2)\Big) + \frac{\theta}{8\pi}(F,F) , \tag{5.70}$$

which is a supersymmetric gauge theory in four dimensions with an $\mathcal{N} = 2$ vector multiplet coupled to an $\mathcal{N} = 2$ hypermultiplet and has underlying gauge Lie algebra $\mathfrak{su}(n)$.

Analogous reductions are clearly possible for the 4-term L_∞-algebras $\widehat{\mathfrak{string}}(\mathfrak{g})$ with $\mathfrak{g}$ any other Lie algebra of type D or E. The four-dimensional theory will then have gauge Lie algebra $\mathfrak{g}$.

It is not too surprising that we are able to reduce our model to super Yang–Mills theory in four dimensions because it contains a free vector multiplet in six dimensions which we reduce to four dimensions. Note that this diverges from what happens in the reduction of the abelian theory: here, an abelian 2-form curvature $\mathcal{F}$ arises from the abelian 3-form curvature $\mathcal{H}$ in the dimensional reduction and both the θ-term and the imaginary part of the modular parameter appear in the reduction of the abelian PST Lagrangian, see [113, 116, 119]. Furthermore, the $\mathsf{SL}(2,\mathbb{Z})$-duality is exhibited by different choices for the auxiliary PST field v, again see [113, 119]. It is, however, not clear how our reduction is related to this when moving to the abelian case. This question is related to the fact that our model contains unwanted Yang–Mills-like degrees of freedom, which should be fixed by a dynamical principle, and would be interesting to study further in the future. In any case, the fact that this reduction is compatible with the supersymmetry mixing the vector and tensor multiplets and that it reproduces the θ-term is pleasing.

5.9 Reduction to supersymmetric Chern–Simons-matter theories

There is no direct argument within M-theory that an effective description of M5-branes should be reducible to one of M2-branes. However, in [30] a model for two M5-branes was shown to reduce to a system of two M2-branes, see also [120]. Moreover, the fact that M2-branes can end on M5-branes has led to attempts of constructing M5-brane models from the M2-brane models, see e.g. [27, 28], which again suggests a link between M5-brane and M2-brane models. Finally, note that while the M2-brane models seem very different from the M5-brane models, the former can be recast in the form of a higher gauge theory [44].

We start from our model (5.33) for L_∞-algebra $\widehat{\mathfrak{string}}_{\mathrm{ext}}(\mathfrak{u}(n)\times\mathfrak{u}(n))$. We choose a metric of split signature on $\mathfrak{u}(n)\times\mathfrak{u}(n)$, anticipating this to become the gauge Lie algebra of the M2-brane model. We then compactify $\mathbb{R}^{1,5}$ to $\mathbb{R}^{1,2}\times S^3$, but a more general choice of compact 3-dimensional spin manifold M^3 should also suffice.

The general dimensional reduction will yield a rather general deformation of the ABJM model. For simplicity, we shall restrict the fields rather severely. While this reduces the supersymmetry of the model, it makes the interpretation of the resulting action clearer. We decompose the fields (B,ϕ,χ) in the tensor multiplets taking values in $\mathbb{R}_r[1]\oplus\mathbb{R}^*_s[1]$ as $\phi=\phi_r+\phi_s$, etc. We then restrict to

$$B_r=0\ ,\quad \phi_r=0\ ,\quad \chi_r=0\ . \tag{5.71}$$

Also, B_s is the connection for a gerbe over S^3 with Dixmier–Douady class k such that

$$\int_{S^3} \mathrm{d}B_s = \frac{k}{2\pi} \tag{5.72}$$

and B_s has no further components. We also restrict the gauge potential such that its components $A_{3,4,5}$ along S^3 vanish. Correspondingly, we demand that the spinors satisfy $\iota_{\frac{\partial}{\partial x^{0,1,2}}} * \bar{\lambda}\gamma_{(3)}\lambda = 0$. With these constraints, the kinematical term for the 3-form curvature reduces according to

$$\begin{aligned} -\tfrac{1}{2}\int_{S^3} \langle \mathcal{H}, *\mathcal{H}\rangle &= -\tfrac{1}{2}\int_{S^3} 2\langle \mathrm{d}B_s, -(A,\mathrm{d}A) - \tfrac{1}{3}(A,[A,A])\rangle \\ &= \frac{k}{2\pi}(A,\mathrm{d}A) + \tfrac{1}{3}(A,[A,A]) \ . \end{aligned} \tag{5.73}$$

We thus obtain the Lagrangian for Chern–Simons theory, and the quantized coupling constant arises from the topological class describing the gerbe over the compactifying 3-manifold M^3.

Let us also consider the PST terms in the action (5.33). It makes sense to restrict the non-vanishing vector field V to be a section of the tangent bundle of $\mathbb{R}^{1,2}$. Then

$$\tfrac{1}{2}\big\langle \iota_V \mathscr{H}, \mathscr{H}\big\rangle \wedge v = \frac{k}{\pi}\mathrm{cs}(A) - \frac{k}{\pi}(\bar{\lambda}, *\gamma_{345}\lambda) \ , \tag{5.74}$$

and we merely get a further contribution to the Chern–Simons term. Altogether, we obtain a supersymmetric Chern–Simons-matter theory coupled to an additional Yang–Mills component with coupling constant ϕ_s, just as in the last section. Again, it makes sense to set $\phi_s = \frac{1}{4g^2_{\mathrm{YM}}}$ to obtain an interacting Chern–Simons-matter theory.

Chapter 6

Conclusions and Outlook

In this thesis, we have argued that higher gauge theory is a useful tool that can give insight into questions arising in string and M-theory. We explicitly constructed two examples of relevance: a non-abelian self-dual string and a six-dimensional (1,0) superconformal field theory. To that end, we discussed the framework of higher gauge theory in Chapter 3 based on the tools introduced in Chapter 2. We also reviewed a generalization of higher gauge theory that was first given in [49].

For the non-abelian self-dual string we considered a higher gauge theory based on two different models of the string Lie 2-algebra — the finite-dimensional skeletal model $\mathfrak{string}_{\rm sk}(\mathfrak{g})$ and the strict loop model $\mathfrak{string}_{\hat\Omega}(\mathfrak{g})$. We discovered that in order to write down a consistent set of equations one needs to modify this by using the twisted skeletal string algebra $\widehat{\mathfrak{string}}_{\rm sk}(\mathfrak{g})$, see (3.47). We also extended this example to the twisted loop model $\widehat{\mathfrak{string}}_{\hat\Omega}(\mathfrak{g})$, see (3.51). In both cases we gave the equations of motion for a non-abelian self-dual string based on a natural generalization of the Bogomolny equation for monopoles. In the skeletal case, this results in (4.38) and (4.40), while the loop case equations are given in (4.46). Both of these can be reduced to the monopole equations in three dimensions. We also gave the elementary non-abelian self-dual string solution in Section 4.7, which satisfies the basic consistency checks. That is, the resulting fields are non-singular over $\mathbb{R}^4$ and interacting in the sense that non-trivial linear combinations of this elementary solution are no longer solutions of the equations of motion. Furthermore, at infinity, the solution approaches the abelian self-dual string in analogy with the behavior of the Dirac and 't Hooft–Polyakov monopoles.

In Chapter 5 we extended the gauge structure based on $\widehat{\mathfrak{string}_{\rm sk}}(\mathfrak{g})$, which allowed us to use it as the underlying structure of the (1,0)-model proposed in [47]. Additionally, this allowed for the inclusion of hypermultiplets and PST-like mechanism in the Lagrangian piecing together steps already considered in [55] and [56]. This led to the Lagrangian given in (5.33) and after implementation of the PST-mechanism to the equations of motion in (5.60), (5.61) and (5.62). This resulting model has many of the properties expected of the (2,0)-theory as discussed in Section 1.2. However, there are a number of crucial discrepancies between our model and the (2,0)-theory which make it clear that this is merely a stepping stone towards an M5-brane model.

First, the Yang–Mills multiplet in our model contains independent Yang–Mills-like degrees of freedom which are clearly incompatible with $\mathcal{N} = (2,0)$ supersymmetry. Analogously to the reduction of M2-brane models to D2-branes, one would expect that these degrees are fixed by a dynamical principle [118]. Furthermore, our dimensional reduction to supersymmetric Yang–Mills theory relies on the presence of these extra Yang–Mills like degrees of freedom and does not line up with the story in the abelian case. It would be nice to understand this further.

Second, and related, the moduli space of vacua of our model is not the one expected from a full M5-brane model. In particular, an M5-brane model should be able to capture the process of separating individual or stacks of M5-brane from another stack. In particular, it should be able to describe the Coulomb branch given by a separation of all individual M5-branes from each other. This is a difficult property to model since it is unclear, already at a mathematical level, how an analogue of the branching $\mathsf{U}(n) \to \mathsf{U}(1)^{\times n}$ relevant for D-branes should work in the case of categorified Lie groups.

Third, the PST mechanism as constructed in [56] relies on a non-vanishing scalar ϕ_s in the tensor multiplet. As stated in [55], this seems to be related to the tensionless string phase transition [111, 112], which, however, is absent under certain conditions. This point requires much further exploration within our model.

A fourth big issue is a general problem of the (1,0)-model which is not fixed by our choice of gauge structure: There is still a single scalar field with a wrong sign in its kinetic term in the action, and one should find an interpretation for its appearance or a mechanism for its elimination.

Fifth, the self-dual string is a BPS solution of our model and is expressible in both the skeletal and loop models of the string algebra and one would hope that a physical model based on higher structures is also agnostic with regards to this categorical equivalence. As we argued in Section 5.7, the formulation of the model of [47] is too rigid to allow for this feature.

Nonetheless our model shows that the construction of an interesting, explicit Lagrangian formulation can go much further than previously suspected. The use of higher gauge theory and categorical structures allows for enough freedom to incorporate many of the desired features and can give useful guiding principles in developing explicit constructions. Interestingly, even though many categorical equivalences, that can be quite coarse, appear due to the use of higher structures, our work shows that the particular choice of what equivalences should be considered is more restricted than thought at first glance. Namely, our constructions crucially depend on working with the twisted models $\widehat{\mathfrak{string}}_{\mathrm{sk}}(\mathfrak{g})$ and $\widehat{\mathfrak{string}}_{\hat{\Omega}}(\mathfrak{g})$ suggesting that the right sort of equivalence is the one considered in Section 3.5.

The issue of categorical inequivalence of our model can then also be regarded in a positive light: it can be seen as a hint that the gauge structure of [47] should be relaxed, which raises the question of how exactly this should be done. A plausible option is to replace the trivially extended algebra $\widehat{\mathfrak{string}}_{\mathrm{ext}}(\mathfrak{g})$ used in Chapter 5 by performing the twist as in Section 3.5 for the whole extended algebra — this should be possible both for the skeletal as well as the loop model. One arrives at modified expressions for the higher curvatures which can be used to redo the steps done in constructing the action. This potentially solves some of the above issues and is part of our ongoing work.

A further possibility for future work is the investigation of possible L_∞-algebras that exhibit the necessary structure to model the separating of individual M5-branes from each other. While 2-term L_∞-algebras are too strict, it is feasible that the 3- and 4-term L_∞-algebras naturally appearing in our model might allow for this.

Appendix A

Explicit Formulae for L_∞-algebras

In this appendix we give some explicit formulations for morphisms and 2-morphisms of L_∞-algebras in terms of multi-brackets as well as show explicitly the equivalence of the homotopy Jacobi relations (2.2) to the corresponding co-derivation squaring to zero. Good references for this include [66], [121] and [122].

A.1 From co-derivations to homotopy Jacobi relations

As outlined in Section 2.2 the homotopy Jacobi relations (2.2) of an L_∞-algebra are equivalent to the condition that the corresponding co-derivation squares to zero, see Definition 2.10. Before showing this explicitly, let us illustrate why when we go from L_∞-algebra to corresponding co-algebra we not only shift the degree but also move from graded anti-symmetric to graded symmetric sign conventions. Let s denote the grade-shift as before, let $\wedge_a$ and $\vee$ denote graded anti-symmetric and graded symmetric conventions[1] , respectively, and finally let e and o denote even and odd generators. Recalling that σ is a function of degree 1 and, thus, induces a minus

[1] This is a slight abuse of notation: In the main body of the thesis we have been using $\wedge$ to denote the product in the differential graded algebra view of L_∞-algebras, which has graded symmetric conventions.

sign when commuted with an odd element, we have,

$$
\begin{aligned}
s^{\otimes 2}(e_1 \wedge_a e_2) &= se_1 \vee se_2 = -se_2 \vee se_1 = -s^2(e_2 \wedge_a e_1) \ , \\
s^{\otimes 2}(e \wedge_a o) &= se \vee so = so \vee se = -s^{\otimes 2}(o \wedge_a e) \ , \\
s^{\otimes 2}(o \wedge_a e) &= -so \vee se = -se \vee so = -s^{\otimes 2}(o \wedge_a e) \ , \\
s^{\otimes 2}(o_1 \wedge_a o_2) &= -so_1 \vee so_2 = -so_2 \vee so_1 = s^{\otimes 2}(o_2 \wedge_a o_1) \ ,
\end{aligned}
\tag{A.1}
$$

which shows that in order for the signs to be consistent when commuting we need to switch sign conventions when shifting the degree.

Equipped with this knowledge we can have a closer look at the relationship between the graded anti-symmetric and graded symmetric Koszul signs χ and ϵ:

$$
\begin{aligned}
&\epsilon(\sigma; x_1, \dots, x_n) x_1 \vee \cdots \vee x_n \\
&= x_{\sigma(1)} \vee \cdots \vee x_{\sigma(n)} \\
&= (-1)^{\sum_{k=1}^{n} (n-k)|x_{\sigma(k)}|} s^{\otimes n}(s^{-1}x_{\sigma(1)} \wedge_a \cdots \wedge_a s^{-1}x_{\sigma(n)}) \\
&= (-1)^{\sum_{k=1}^{n} (n-k)|x_{\sigma(k)}|} \chi(\sigma; s^{-1}x_1, \dots, s^{-1}x_n)\, s^{\otimes n}(s^{-1}x_1 \wedge_a \cdots \wedge_a s^{-1}x_n) \\
&= (-1)^{\sum_{k=1}^{n} (n-k)(|x_{\sigma(k)}| + |x_k|)} \chi(\sigma; s^{-1}x_1, \dots, s^{-1}x_n)\, x_1 \vee \cdots \vee x_n \ ,
\end{aligned}
\tag{A.2}
$$

where $|x|$ denotes the degree of x. This implies

$$
\epsilon(\sigma; x_1, \dots, x_n) = (-1)^{\sum_{k=1}^{n} (n-k)\left(\left|x_{\sigma(k)}\right| + |x_k|\right)} \chi(\sigma; s^{-1}x_1, \dots, s^{-1}x_n) \ . \tag{A.3}
$$

Using this and the fact that co-derivations are uniquely determined by their image, so that is enough to consider $\mathcal{D}^1 \circ \mathcal{D} = 0$, we calculate

$$
\begin{aligned}
0 &= \mathcal{D}^1(\mathcal{D}(x_1 \vee \cdots \vee x_n)) \\
&= \sum_{i+j=n} \sum_{\sigma \in S_{i|j}} \epsilon(\sigma; x_1, \ldots, x_n) \mathcal{D}^1(\mathcal{D}^1(x_{\sigma(1)} \vee \cdots \vee x_{\sigma(i)}) \vee \cdots \vee x_{\sigma(n)}) \\
&= \sum_{i+j=n} \sum_{\sigma \in S_{i|j}} (-1)^{\frac{i(i-1)+(j+1)j}{2}} \epsilon(\sigma; x_1, \ldots, x_n) \\
&\quad \cdot s\mu_{j+1}(s^{-1})^{\otimes j+1}(s\mu_i(s^{-1})^{\otimes i}(x_{\sigma(1)} \vee \cdots \vee x_{\sigma(i)}) \vee x_{\sigma(i+1)} \vee \cdots \vee x_{\sigma(n)}) \\
&= \sum_{i+j=n} \sum_{\sigma \in S_{i|j}} (-1)^{\frac{i(i-1)+(j+1)j}{2} - j + \sum\limits_{k=1}^{n} (n-k)|x_{\sigma(k)}|} \epsilon(\sigma; x_1, \ldots, x_n) \\
&\quad \cdot s\mu_{j+1}(s^{-1} s\mu_i(s^{-1}x_{\sigma(1)} \wedge_a \cdots \wedge_a s^{-1}x_{\sigma(i)}) \wedge_a s^{-1}x_{\sigma(i+1)} \wedge_a \cdots \wedge_a s^{-1}x_{\sigma(n)}) \\
&= \sum_{i+j=n} \sum_{\sigma \in S_{i|j}} (-1)^{\frac{i(i-1)+(j+1)j}{2} - j + \sum\limits_{k=1}^{n} (n-k)|x_k|} \chi(\sigma; s^{-1}x_1, \ldots, s^{-1}x_n) \\
&\quad \cdot s\mu_{j+1}(s^{-1} s\mu_i(s^{-1}x_{\sigma(1)} \wedge_a \cdots \wedge_a s^{-1}x_{\sigma(i)}) \wedge_a s^{-1}x_{\sigma(i+1)} \wedge_a \cdots \wedge_a s^{-1}x_{\sigma(n)})
\end{aligned}
\tag{A.4}
$$

where in the first two lines we have made use of relations (2.23) and (2.24). Furthermore, we have

$$
\begin{aligned}
\frac{i(i-1)+(j+1)j}{2} - j &= \frac{i^2 - i + j^2 - j}{2} \\
&= \frac{n^2 - n}{2} - ij \ .
\end{aligned}
\tag{A.5}
$$

As $\frac{n^2-n}{2}$ and $\sum (n-k)\,|x_k|$ are independent of the splitting $i+j=n$ as well as the permutation σ, we can pull the corresponding minus signs out of the sums, which yields

$$
\begin{aligned}
\mathcal{D}^1(\mathcal{D}(x_1 \vee \cdots \vee x_n)) = \pm \sum_{i+j=n} \sum_{\sigma \in S_{i|j}} (-1)^{ij} \chi(\sigma; s^{-1}x_1, \ldots, s^{-1}x_n) \\
\cdot s\mu_{j+1}(\mu_i(s^{-1}x_{\sigma(1)}, \ldots, s^{-1}x_{\sigma(i)}), s^{-1}x_{\sigma(i+1)}, \ldots, s^{-1}x_{\sigma(n)}) \ ,
\end{aligned}
\tag{A.6}
$$

which proves the claim. The analogous fact for Chevalley-Eilenberg algebras, see Definition 2.13, is just the dual version of this calculation.

A.2 Explicit formulae for morphisms of L_∞-algebras

A straightforward way of defining morphisms between L_∞-algebras is via morphisms between the corresponding co-algebras that respect the co-derivation, cf. Definition 2.11. In this section we will investigate what this means in terms of the higher brackets of an L_∞-algebra and give explicit formulae for morphisms of arbitrary n-term L_∞-algebras.

Recall, that such a co-algebra morphism Ψ is completely defined by its restricted image Ψ^1. In order to write down the explicit formula we need the following definitions: Let $\Gamma(n,p)$ be the space of ordered partitions of n into p summands. That is, an element $\vec{\lambda} \in \Gamma(n,p)$ is of the form $\vec{\lambda} = (\lambda_1, \ldots, \lambda_p)$, where $\lambda_1 + \cdots + \lambda_p = n$ and $\lambda_1 \geq \cdots \geq \lambda_p$. An equivalent way of writing the same partition λ is as a vector $(i_1, \ldots, i_n)$, where i_j denotes the number of times j appears in $\vec{\lambda}$. Let us further define $l_{\vec{\lambda}} = i_1! \cdot \cdots \cdot i_n!$, i.e. $l_{\vec{\lambda}}$ is the number of permutations that leave $\vec{\lambda}$ invariant. Furthermore, let $S_{\vec{\lambda}}$ be the set of $(\lambda_1, \ldots, \lambda_p)$-unshuffles, that is, the permutations which map $1, \ldots, n$ to a set of ordered lists of length λ_i. A co-algebra morphism out of $\vee^\bullet(V)$ then acts as follows, cf. [66, Appendix A]:

$$
\begin{aligned}
&\Psi(x_1 \vee \cdots \vee x_n) = \\
&\sum_{p=1}^{n} \sum_{\vec{\lambda} \in \Gamma(n,p)} \sum_{\sigma \in S_{\vec{\lambda}}} \tfrac{\epsilon(\sigma; x_1, \ldots, x_n)}{l_{\vec{\lambda}}} \big(\Psi^1_{\lambda_1} \otimes \cdots \otimes \Psi^1_{\lambda_p}\big)\big(x_{\sigma(1)} \vee \cdots \vee x_{\sigma(n)}\big) \ ,
\end{aligned}
\tag{A.7}
$$

where Ψ^1_i is the part of Ψ^1 that acts on i generators. Similarly to $\mathcal{D}^1$, these correspond to maps ψ_i in the multi-bracket viewpoint, i.e.

$$
\Psi^1_i = (-1)^{\frac{i(i-1)}{2}} s \circ \psi_i \circ (s^{-1})^{\otimes i} \ , \tag{A.8}
$$

where again s is the degree-shift made explicit and ψ_i also acts on i generators.

Equipped with this we can turn our attention to the condition $\Psi \circ \mathcal{D} = \mathcal{D} \circ \Psi$, where, as before, it is sufficient to consider $\Psi^1 \circ \mathcal{D} = \mathcal{D}^1 \circ \Psi$ only. By the same calculation as in (A.4) we have

$$\Psi^1 \circ \mathcal{D}(x_1 \vee \cdots \vee x_n)$$

$$= (-1)^{\frac{n(n-1)}{2} + \sum\limits_{k=1}^{n}(n-k)|x_k|} \sum_{i+j=n} \sum_{\sigma \in S_{i|j}} (-1)^{ij} \chi(\sigma; s^{-1}x_1, \ldots, s^{-1}x_n) \tag{A.9}$$

$$\cdot\, s\psi_{j+1}(\mu_i(s^{-1}x_{\sigma(1)} \wedge_a \cdots \wedge_a s^{-1}x_{\sigma(i)}) \wedge_a s^{-1}x_{\sigma(i+1)} \wedge_a \cdots \wedge_a s^{-1}x_{\sigma(n)}) \ .$$

Similarly, we get

$$\mathcal{D}^1 \circ \Psi(x_1 \vee \cdots \vee x_n)$$

$$= \sum_{p=1}^{n} \sum_{\vec{\lambda} \in \Gamma(n,p)} \sum_{\sigma \in S_{\vec{\lambda}}} \tfrac{\epsilon(\sigma; x_1, \ldots, x_n)}{l_{\vec{\lambda}}} \mathcal{D}^1\big((\Psi^1_{\lambda_1} \otimes \cdots \otimes \Psi^1_{\lambda_p})(x_{\sigma(1)} \vee \cdots \vee x_{\sigma(n)})\big)$$

$$= \sum_{p=1}^{n} \sum_{\vec{\lambda} \in \Gamma(n,p)} \sum_{\sigma \in S_{\vec{\lambda}}} (-1)^{\frac{p(p-1)}{2} + \sum\limits_{k=1}^{n}(n-k)|x_{\sigma(k)}| + \sum\limits_{i=1}^{p} \frac{\lambda_i(\lambda_i-1)}{2}} \frac{\epsilon(\sigma; x_1, \ldots, x_n)}{l_{\vec{\lambda}}}$$

$$\cdot\, s\mu_p(\psi_{\lambda_1}(s^{-1}x_1 \wedge_a \cdots \wedge_a s^{-1}x_{\lambda_1}) \wedge_a \cdots \wedge_a \psi_{\lambda_p}(s^{-1}x_{n-\lambda_p+1} \wedge_a \cdots \wedge_a s^{-1}x_n))$$

$$= \sum_{p=1}^{n} \sum_{\vec{\lambda} \in \Gamma(n,p)} \sum_{\sigma \in S_{\vec{\lambda}}} (-1)^{\frac{p(p-1)}{2} + \sum\limits_{k=1}^{n}(n-k)|x_k)| + \sum\limits_{i=1}^{p} \frac{\lambda_i(\lambda_i-1)}{2}} \frac{\chi(\sigma; s^{-1}x_1, \ldots, s^{-1}x_n)}{l_{\vec{\lambda}}}$$

$$\cdot\, s\mu_p(\psi_{\lambda_1}(s^{-1}x_1 \wedge_a \cdots \wedge_a s^{-1}x_{\lambda_1}) \wedge_a \cdots \wedge_a \psi_{\lambda_p}(s^{-1}x_{n-\lambda_p+1} \wedge_a \cdots \wedge_a s^{-1}x_n)) \ . \tag{A.10}$$

Then using

$$\sum_{i=1}^{p} \frac{\lambda_i(\lambda_i - 1)}{2} = \frac{n(n-1)}{2} - \sum_{i>j} \lambda_i \lambda_j \ , \tag{A.11}$$

we arrive at a general formula for morphisms of L_∞-algebras in terms of multi-brackets:

$$\sum_{i+j=n}\sum_{\sigma\in S_{i|j}}(-1)^{ij}\chi(\sigma;x_1,\dots,x_n)\psi_{j+1}(\mu_i(x_{\sigma(1)},\dots,x_{\sigma(i)}),\dots,x_{\sigma(n)})$$

$$=\sum_{p=1}^{n}\sum_{\vec{\lambda}\in\Gamma(n,p)}\sum_{\sigma\in S_{\vec{\lambda}}}(-1)^{\frac{p(p-1)}{2}+\sum_{i>j}\lambda_i\lambda_j}\frac{\chi(\sigma;x_1,\dots,x_n)}{l_{\vec{\lambda}}}\mu_p(\psi_{\lambda_1}(x_1,\dots,x_{\lambda_1}),\tag{A.12}$$

$$\dots,\psi_{\lambda_p}(x_{n-\lambda_p+1},\dots,x_n))\ .$$

To illuminate this, let us spell out the first few identities, i.e. for $n=1,2$ and 3, which leads to

$$0=\psi_1(\mu_1(x_1))-\mu_1(\psi_1(x_1))\ ,$$

$$0=\psi_1(\mu_2(x_1,x_2))-\psi_2(\mu_1(x_1),x_2)+(-1)^{|x_1||x_2|}\psi_2(\mu_1(x_2),x_1)$$

$$-\mu_1(\psi_2(x_1,x_2))-\mu_2(\psi_1(x_1),\psi_1(x_2))\ ,$$

$$0=\psi_1(\mu_3(x_1,x_2,x_3))+\psi_2(\mu_2(x_1,x_2),x_3)-(-1)^{|x_2||x_3|}\psi_2(\mu_2(x_1,x_3),x_2)$$

$$+(-1)^{|x_1|(|x_2|+|x_3|)}\psi_2(\mu_2(x_2,x_3),x_1)+\psi_3(\mu_1(x_1),x_2,x_3)\tag{A.13}$$

$$-(-1)^{|x_1||x_2|}\psi_3(\mu_1(x_2),x_1,x_3)+(-1)^{|x_3|(|x_1|+|x_2|)}\psi_3(\mu_1(x_3),x_1,x_2)$$

$$-\mu_1(\psi_3(x_1,x_2,x_3))+\mu_2(\psi_2(x_1,x_2),\psi_1(x_3))-(-1)^{|x_2||x_3|}$$

$$\mu_2(\psi_2(x_1,x_3),\psi_1(x_2))+(-1)^{|x_1|(|x_2|+|x_3|)}\mu_2(\psi_2(x_2,x_3),\psi_1(x_1))$$

$$-\mu_3(\psi_1(x_1),\psi_1(x_2),\psi_3(x_3))\ .$$

For a 2-term L_∞-algebra with generators x and r of degree 0 and 1, respectively, this translates to the four identities

$$
\begin{aligned}
0 &= \psi_1(\mu_1(r)) - \mu_1(\psi_1(r)) \ , \\
0 &= \psi_1(\mu_2(x_1,x_2)) - \mu_1(\psi_2(x_1,x_2)) - \mu_2(\psi_1(x_1),\psi_1(x_2)) \ , \\
0 &= \psi_1(\mu_2(x,r)) + \psi_2(\mu_1(r),x) - \mu_2(\psi_1(x),\psi_1(r)) \ , \\
0 &= \psi_1(\mu_3(x_1,x_2,x_3)) + \psi_2(\mu_2(x_1,x_2),x_3) - \psi_2(\mu_2(x_1,x_3),x_2) \\
&\quad + \psi_2(\mu_2(x_2,x_3),x_1) - \mu_3(\psi_1(x_1),\psi_1(x_2),\psi_1(x_3)) \\
&\quad + \mu_2(\psi_2(x_1,x_2),\psi_1(x_3)) - \mu_2(\psi_2(x_1,x_3),\psi_1(x_2)) \\
&\quad + \mu_2(\psi_2(x_2,x_3),\psi_1(x_1)) \ ,
\end{aligned}
\tag{A.14}
$$

which is in complete agreement with (2.29).

A.3 Explicit formulae for 2-morphisms of L_∞-algebras

In Section 2.4 we introduced an explicit notion for 2-morphisms of L_∞-algebras, see Definition 2.19. In this thesis, we use these 2-morphisms for up to 3-term L_∞-algebras and therefore collect the corresponding formulas here, explicitly. Thus, let $\mathfrak{g}$ and $\mathfrak{h}$ be 3-term L_∞-algebras with generators t^α, b^a, c^μ and t'^α, b'^a, c'^μ of degree 1, 2, and 3, respectively. Let $\Phi, \Psi : \mathsf{CE}(\mathfrak{g}) \to \mathsf{CE}(\mathfrak{h})$ be morphisms between their Chevalley-Eilenberg. A generic differential for $\mathsf{CE}(\mathfrak{g})$ is given by

$$
\begin{aligned}
Qt^\alpha &= -\tfrac{1}{2} f^\alpha_{\beta\gamma} t^\beta \wedge t^\gamma - f^\alpha_a b^a \ , \\
Qb^a &= -\tfrac{1}{3!} f^a_{\alpha\beta\gamma} t^\alpha \wedge t^\beta \wedge t^\gamma - f^a_{\alpha b} t^\alpha \wedge b^b - f^a_\mu c^\mu \ , \\
Qc^\mu &= -\tfrac{1}{4!} f^\mu_{\alpha\beta\gamma\delta} t^\alpha \wedge t^\beta \wedge t^\gamma \wedge t^\delta - \tfrac{1}{2} f^\mu_{\alpha\beta a} t^\alpha \wedge t^\beta \wedge b^a - f^\mu_{\alpha\nu} t^\alpha c^\nu - \tfrac{1}{2} f^\mu_{ab} b^a \wedge b^b \ ,
\end{aligned}
\tag{A.15}
$$

with analogous expressions for $\mathsf{CE}(\mathfrak{h})$. The maps Φ and Ψ are given on generators by

$$\begin{aligned} \Phi t^\alpha &= \Phi^\alpha_\beta t'^\beta, \qquad \Phi b^a = \Phi^a_b b'^b + \tfrac{1}{2}\Phi^a_{\alpha\beta} t'^\alpha \wedge t'^\beta \ , \\ \Psi t^\alpha &= \Psi^\alpha_\beta t'^\beta, \qquad \Psi b^a = \Psi^a_b b'^b + \tfrac{1}{2}\Psi^a_{\alpha\beta} t'^\alpha \wedge t'^\beta \ , \end{aligned} \tag{A.16}$$

and

$$\begin{aligned} \Phi c^\mu &= \Phi^\mu_\nu c'^\nu + \Phi^\mu_{\alpha a} t'^\alpha \wedge b'^a + \tfrac{1}{3!}\Phi^\mu_{\alpha\beta\gamma} t'^\alpha \wedge t'^\beta \wedge t'^\gamma \ , \\ \Psi c^\mu &= \Psi^\mu_\nu c'^\nu + \Psi^\mu_{\alpha a} t'^\alpha \wedge b'^a + \tfrac{1}{3!}\Psi^\mu_{\alpha\beta\gamma} t'^\alpha \wedge t'^\beta \wedge t'^\gamma \ . \end{aligned} \tag{A.17}$$

Lastly, the degree -1 map η can generically be written as

$$\begin{aligned} \eta(t^\alpha) &= 0 \ , \\ \eta(b^a) &= \eta^a_\alpha t'^\alpha \ , \\ \eta(c^\mu) &= \eta^\mu_a b'^a + \tfrac{1}{2}\eta^\mu_{\alpha\beta} t'^\alpha \wedge t'^\beta \ . \end{aligned} \tag{A.18}$$

With this we can use formula (2.54) and the requirement that η vanishes along $\mathfrak{g}^*[2] \subset \mathsf{W}(\mathfrak{g})$ to calculate

$$\begin{aligned} \eta(Qt^\alpha) &= -f^\alpha_a \eta^a_\beta t'^\beta \ , \\ \eta(Qb^a) &= \tfrac{1}{2} f^a_{\alpha b}\eta^b_\gamma(\Psi^\alpha_\beta + \Phi^\alpha_\beta) t'^\beta \wedge t'^\gamma - f^a_\mu \eta^\mu_b b'^b - \tfrac{1}{2} f^a_\mu \eta^\mu_{\alpha\beta} t'^\alpha \wedge t'^\beta \ , \\ \eta(Qc^\mu) &= -\tfrac{1}{3!} f^\mu_{\alpha\beta a}\eta^a_\epsilon(\Psi^\alpha_\gamma\Psi^\beta_\delta + \Psi^\alpha_\gamma\Phi^\beta_\delta + \Phi^\alpha_\gamma\Phi^\beta_\delta) t'^\gamma \wedge t'^\delta \wedge t'^\epsilon \\ &\quad + \tfrac{1}{2} f^\mu_{\alpha\nu}\eta^\nu_a(\Psi^\alpha_\beta + \Phi^\alpha_\beta) t'^\beta \wedge b'^a + \tfrac{1}{4} f^\mu_{\alpha\nu}\eta^\nu_{\gamma\delta}(\Psi^\alpha_\beta + \Phi^\alpha_\beta) t'^\beta \wedge t'^\gamma \wedge t'^\delta \\ &\quad - \tfrac{1}{2} f^\mu_{ab}\eta^b_\gamma(\Psi^a_c + \Phi^a_c) t'^\gamma \wedge b'^c - \tfrac{1}{4} f^\mu_{ab}\eta^b_\gamma(\Psi^a_{\alpha\beta} + \Phi^a_{\alpha\beta}) t'^\alpha \wedge t'^\beta \wedge t'^\gamma \ . \end{aligned} \tag{A.19}$$

This consequently leads to

$$\begin{aligned} [Q,\eta] t^\alpha &= -f^\alpha_a \eta^a_\beta t'^\beta \ , \\ [Q,\eta] b^a &= -\tfrac{1}{2}\eta^a_\alpha f'^\alpha_{\beta\gamma} t'^\beta \wedge t'^\gamma - \eta^a_\alpha f'^\alpha_b b'^b + \tfrac{1}{2} f^a_{\alpha b}\eta^b_\gamma(\Psi^\alpha_\beta + \Phi^\alpha_\beta) t'^\beta \wedge t'^\gamma \\ &\quad - f^a_\mu \eta^\mu_b b'^b - \tfrac{1}{2} f^a_\mu \eta^\mu_{\alpha\beta} t'^\alpha \wedge t'^\beta \ , \end{aligned} \tag{A.20}$$

and

$$
\begin{aligned}
[Q,\eta]c^\mu = &-\tfrac{1}{3!}\eta^\mu_a f'^a_{\alpha\beta\gamma}t'^\alpha\wedge t'^\beta\wedge t'^\gamma - \eta^\mu_a f'^a_{\alpha b}t'^\alpha\wedge b'^b - \eta^\mu_a f'^a_\nu c'^\nu \\
&- \tfrac{1}{2}\eta^\mu_{\alpha\beta}f'^\beta_{\gamma\delta}t'^\alpha\wedge t'^\gamma\wedge t'^\delta - \eta^\mu_{\alpha\beta}f'^\beta_a t'^\alpha\wedge b'^b \\
&- \tfrac{1}{3!}f^\mu_{\alpha\beta a}\eta^a_\epsilon(\Psi^\alpha_\gamma\Psi^\beta_\delta+\Psi^\alpha_\gamma\Phi^\beta_\delta+\Phi^\alpha_\gamma\Phi^\beta_\delta)t'^\gamma\wedge t'^\delta\wedge t'^\epsilon \\
&+ \tfrac{1}{2}f^\mu_{\alpha\nu}\eta^\nu_a(\Psi^\alpha_\beta+\Phi^\alpha_\beta)t'^\beta\wedge b'^a + \tfrac{1}{4}f^\mu_{\alpha\nu}\eta^\nu_{\gamma\delta}(\Psi^\alpha_\beta+\Phi^\alpha_\beta)t'^\beta\wedge t'^\gamma\wedge t'^\delta \\
&- \tfrac{1}{2}f^\mu_{ab}\eta^b_\gamma(\Psi^a_c+\Phi^a_c)t'^\gamma\wedge b'^c - \tfrac{1}{4}f^\mu_{ab}\eta^b_\gamma(\Psi^a_{\alpha\beta}+\Phi^a_{\alpha\beta})t'^\alpha\wedge t'^\beta\wedge t'^\gamma\ .
\end{aligned} \tag{A.21}
$$

Thus, the identity $\Phi-\Psi=[Q,\eta]$ finally gives rise to the following conditions for 2-morphisms in the case of 3-term L_∞-algebras:

$$
\begin{aligned}
\Phi^\alpha_\beta - \Psi^\alpha_\beta &= -f^\alpha_a\eta^a_\beta\ , \\
\Phi^a_b - \Psi^a_b &= -\eta^a_\alpha f'^\alpha_b - f^a_\mu\eta^\mu_b\ , \\
\Phi^a_{[\beta\gamma]} - \Psi^a_{[\beta\gamma]} &= -\eta^a_\alpha f'^\alpha_{[\beta\gamma]} + f^a_{\alpha b}(\Psi+\Phi)^\alpha_{[\beta}\eta^b_{\gamma]} - f^a_\mu\eta^\mu_{[\beta\gamma]}\ , \\
\Phi^\mu_\nu - \Psi^\mu_\nu &= -\eta^\mu_a f'^a_\nu\ , \\
\Phi^\mu_{\beta c} - \Psi^\mu_{\beta c} &= -\eta^\mu_a f'^a_{\beta c} - \eta^\mu_{\beta\alpha}f'^\alpha_c + \tfrac{1}{2}f^\mu_{\alpha\nu}(\Psi+\Phi)^\alpha_\beta\eta^\nu_c - \tfrac{1}{2}f^\mu_{\alpha b}(\Psi+\Phi)^a_c\eta^b_\beta\ , \\
\Phi^\mu_{[\gamma\delta\epsilon]} - \Psi^\mu_{[\gamma\delta\epsilon]} &= -\eta^\mu_a f'^a_{[\gamma\delta\epsilon]} + 3\eta^\mu_{\alpha[\gamma}f'^\alpha_{\delta\epsilon]} - f^\mu_{\alpha\beta a}(\Psi\Psi+\Psi\Phi+\Phi\Phi)^{\alpha\beta}_{[\gamma\delta}\eta^a_{\epsilon]} \\
&\quad + \tfrac{2}{3}f^\mu_{\alpha\nu}(\Psi+\Phi)^\alpha_{[\gamma}\eta^\nu_{\delta\epsilon]} - \tfrac{2}{3}f^\mu_{ab}(\Psi+\Phi)^a_{[\gamma\delta}\eta^b_{\epsilon]}\ .
\end{aligned} \tag{A.22}
$$

Appendix B

Calculations for the 6d (1,0) Superconformal Model

In this appendix we give a detailed calculation of the Lagrangian variation of the model in Chapter 5. In B.1, we start by giving the space-time and spinor conventions and a few identities crucial to the calculations before giving the full variation and symmetries of the Lagrangian in B.2.

B.1 Conventions and identities

We use the same conventions as given in [47, Appendix A] and repeat these here for convenience.

We work on flat six-dimensional space-time $\mathbb{R}^{1,5}$ with mostly positive metric and and Levi–Civita tensor $\varepsilon_{012345} = 1$. We have the six dimensional gamma matrices γ_μ satisfying

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu} \ , \tag{B.1}$$

together with the chirality matrix $\gamma_y = \gamma_1 \cdots \gamma_5$. The chiralities of the fermions and supersymmetry transformation parameter are given by

$$\gamma_7\lambda^i = \lambda^i \ , \quad \gamma_7\chi^i = -\chi^i \qquad \text{and} \qquad \gamma_7\varepsilon^i = \varepsilon^i \ . \tag{B.2}$$

Furthermore, we use the notation $\gamma_{(p)} = \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}\, \gamma_{\mu_1} \ldots \gamma_{\mu_p}$.

The fermions carry $\mathsf{Sp}(1)$ indices i, j which are raised and lowered using ε^{ij} and its inverse ε_{ij}: $\lambda_i = \varepsilon_{ij}\lambda^j$ and $\lambda^i = \varepsilon^{ij}\lambda_j$. Also, we use the usual NW-SE contractions of indices whenever unambiguous: $\bar{\lambda}\psi = \bar{\lambda}^i\psi_i$. The products of fermions and gamma matrices then satisfy the following symmetry properties

$$\bar{\lambda}\gamma_{(p)}\chi = t_p\bar{\chi}\gamma_{(p)}\lambda \ , \quad t_p = \begin{cases} +1: & p = 0,3,4 \ , \\ -1: & p = 1,2,5,6 \ . \end{cases} \tag{B.3}$$

Before giving the full Lagrangian variation let us list a few identities that will frequently prove useful in all subsequent calculations. First, note that using the non-vanishing non-null one-form v and its dual vector field V as given in (5.40), we can write any p-form $\omega_{(p)} \in \Omega^p(\mathbb{R}^{1,5})$ as $\omega_{(p)} = v \wedge \alpha + *(v \wedge \beta)$. This implies the very useful identity

$$\omega_{(p)} = (-1)^{p+1}(\iota_V\omega_{(p)}) \wedge v + *((\iota_V * \omega_{(p)}) \wedge v) \ , \tag{B.4}$$

which can be seen from

$$(-1)^{p+1}(\iota_V\omega_{(p)}) \wedge v + *((\iota_V * \omega_{(p)}) \wedge v) = (-1)^{p+1}\alpha \wedge v + (-1)^{p+1} * (\beta \wedge v) \ . \tag{B.5}$$

For $p = 6$, this identity reduces to

$$\omega_{(6)} = -(\iota_V\omega_{(6)}) \wedge v \ . \tag{B.6}$$

Moreover, a direct computation shows that for any $\omega_{(p)} \in \Omega^3(\mathbb{R}^{1,5})$,

$$\iota_V * \omega_{(p)} = *(\omega_{(p)} \wedge v) \ . \tag{B.7}$$

It is also useful to recall the standard identity

$$\omega_{(p)} \wedge *\eta_{(p)} = \eta_{(p)} \wedge *\omega_{(p)} \ , \tag{B.8}$$

for p-forms $\omega_{(p)}$ and $\eta_{(p)}$ in $\Omega^p(\mathbb{R}^{1,5})$. Lastly, let us note that

$$\langle\kappa(\bar{\lambda}, \gamma_{(3)}\lambda), \kappa(\bar{\lambda}, \gamma_{(3)}\lambda)\rangle = 0 \tag{B.9}$$

in $\widehat{\mathfrak{string}}_{\rm ext}(\mathfrak{g})$. This is essential for the PST mechanism to extend to the supersymmetric case and is not satisfied in the general case of the gauge structure of [47].

B.2 Full Lagrangian variation and symmetries

Equipped with the above identities let us start by calculating the general variations of the curvatures. In all of these we repeatedly use the homotopy Jacobi relations of our gauge L_∞-algebra $\widehat{\mathfrak{string}}_{\rm ext}(\mathfrak{g})$. For the two-form curvature $\mathcal{F}$ we straightforwardly compute

$$\begin{aligned} \delta\mathcal{F} &= {\rm d}\delta A + \mu_2(A, \delta A) + \mu_1(\delta B) \\ &= \nabla\delta A + \mu_1(\Delta B)\ . \end{aligned} \tag{B.10}$$

We continue with $\mathcal{H}$ for which we have

$$\begin{aligned} \delta\mathcal{H} &= {\rm d}\delta B - \kappa(\delta A, {\rm d}A) - \kappa(A, {\rm d}\delta A) - \kappa(\delta A, \mu_2(A,A)) + \mu_1(\delta C) \\ &= {\rm d}(\delta B + \kappa(A, \delta A)) - 2\kappa({\rm d}A, \delta A) - \kappa(\mu_2(A,A), \delta A) + \mu_1(\Delta C) \\ &= {\rm d}(\Delta B) - \kappa(2{\rm d}A + \mu_2(A,A) + 2\mu_1(B), \delta A) + \mu_1(C) \\ &= {\rm d}(\Delta B) - 2\kappa(\mathcal{F}, \delta A) + \mu_1(\Delta C)\ . \end{aligned} \tag{B.11}$$

Furthermore, for $\mathcal{G}$ we obtain

$$\begin{aligned} \delta\mathcal{G} &= {\rm d}\delta C + \mu_2(A, \delta C) + \mu_2(\delta A, C) + \kappa(\delta\mathcal{F}, B) + \kappa(\mathcal{F}, \delta B) + \mu_1(\delta D) \\ &= \nabla\delta C + \mu_2(\delta A, C) + \kappa(D\delta A, B) + \kappa(\mathcal{F}, \Delta B) + \mu_1(\delta D) \\ &\quad + \kappa(\delta A, {\rm d}B) - \kappa(\delta A, {\rm d}B) \\ &= \nabla\delta C - \kappa(\delta A, {\rm d}B) + \kappa(D\delta A, B) + \kappa(\delta A, \mathcal{H}) + \kappa(\mathcal{F}, \Delta B) + \mu_1(\Delta D) \\ &= \nabla(\Delta C) + \kappa(\delta A, \mathcal{H}) + \kappa(\mathcal{F}, \Delta B) + \mu_1(\Delta D)\ . \end{aligned} \tag{B.12}$$

The variation for $\mathcal{I}$ follows directly and, altogether, we reproduce the expressions given in (5.17).

Equipped with these we can consider the variation of the Lagrangian given in (5.33). In the following we will show the crucial parts of the calculation leading to equation (5.45) in detail. We then list the full variation while skipping over the details of the remaining calculations, as these are straightforward.

Let us start with the terms in $\mathcal{L}_{\text{PST}}$ and $\mathcal{L}_{\text{tensor}}$ containing $\mathcal{H}$, which leads to

$$\begin{aligned}
(\mathcal{L}'_{\text{PST}})|_{\mathcal{H}} &= (\mathcal{L}_{\text{PST}} + \mathcal{L}_{\text{tensor}})|_{\mathcal{H}} \\
&= \tfrac{1}{2}\big\langle \iota_V \mathscr{H}, \mathscr{H}\big\rangle \wedge v - \tfrac{1}{2}\big\langle \mathcal{H}, *\mathcal{H}\big\rangle + \big\langle \mathcal{H}, \kappa(\bar{\lambda}, \gamma_{(3)}\lambda)\big\rangle \\
&\overset{\text{(B.9)}}{=} \tfrac{1}{2}\big\langle \iota_V * \mathcal{H}, *\mathcal{H}\big\rangle \wedge v - \tfrac{1}{2}\big\langle \iota_V * \mathcal{H}, \mathcal{H}\big\rangle \wedge v - \tfrac{1}{2}\big\langle \iota_V * \mathcal{H}, \kappa(\bar{\lambda}, \gamma_{(3)}\lambda)\big\rangle \wedge v \\
&\quad - \tfrac{1}{2}\big\langle \iota_V \mathcal{H}, *\mathcal{H}\big\rangle \wedge v + \tfrac{1}{2}\big\langle \iota_V \mathcal{H}, \mathcal{H}\big\rangle \wedge v + \tfrac{1}{2}\big\langle \iota_V \mathcal{H}, \kappa(\bar{\lambda}, \gamma_{(3)}\lambda)\big\rangle \wedge v \\
&\quad - \tfrac{1}{2}\big\langle \iota_V \kappa(\bar{\lambda}, \gamma_{(3)}\lambda), *\mathcal{H} - \mathcal{H}\big\rangle \wedge v - \tfrac{1}{2}\big\langle \mathcal{H}, *\mathcal{H}\big\rangle + \big\langle \mathcal{H}, \kappa(\bar{\lambda}, \gamma_{(3)}\lambda)\big\rangle \\
&\underset{\text{(B.8)}}{\overset{\text{(B.7)}}{=}} \big\langle \iota_V \mathcal{H}, \mathcal{H}\big\rangle \wedge v - \tfrac{1}{2}\big\langle \iota_V * \mathcal{H}, \mathcal{H}\big\rangle \wedge v - \tfrac{1}{2}\big\langle * (\iota_V \mathcal{H} \wedge v), \mathcal{H}\big\rangle \\
&\quad + \big\langle \iota_V \kappa(\bar{\lambda}, \gamma_{(3)}\lambda), \mathcal{H}\big\rangle \wedge v - \big\langle * (\iota_V \kappa(\bar{\lambda}, \gamma_{(3)}\lambda) \wedge v), \mathcal{H}\big\rangle - \tfrac{1}{2}\big\langle \mathcal{H}, *\mathcal{H}\big\rangle \\
&\quad + \big\langle \mathcal{H}, \kappa(\bar{\lambda}, \gamma_{(3)}\lambda)\big\rangle \\
&\overset{\text{(B.4)}}{=} -\big\langle \iota_V(*\mathcal{H} - \mathcal{H} - 2\kappa(\bar{\lambda}, \gamma_{(3)}\lambda)), \mathcal{H}\big\rangle \wedge v \ ,
\end{aligned}$$

which is in agreement with (5.41). We move on to calculating the variation of this term suppressing the variation with respect to λ. We obtain

$$\begin{aligned}
\delta(\mathcal{L}'_{\text{PST}})|_{\mathcal{H}} &\underset{\text{(B.8)}}{\overset{\text{(B.7)}}{=}} -\big\langle (\delta\mathcal{H} - *\delta\mathcal{H}) \wedge v, \iota_V * \mathcal{H}\big\rangle - \big\langle \iota_V(*\mathcal{H} - \mathcal{H} - 2\kappa(\bar{\lambda}, \gamma_{(3)}\lambda)), \delta\mathcal{H}\big\rangle \wedge v \\
&\quad + \Big[\big\langle \iota_V * \mathcal{H}, *\mathcal{H} - \mathcal{H} - 2\kappa(\bar{\lambda}, \gamma_{(3)}\lambda)\big\rangle - \big\langle \iota_V(*\mathcal{H} - \mathcal{H} - 2\kappa(\bar{\lambda}, \gamma_{(3)}\lambda)), \\
&\qquad \mathcal{H}\big\rangle\Big] \wedge \delta v + \delta_\lambda(\mathcal{L}'_{\text{PST}})|_{\mathcal{H}} \\
&\overset{\text{(B.6)}}{=} 2\big\langle \iota_V \mathscr{H} \wedge v, \delta\mathcal{H}\big\rangle + \big\langle \iota_V \mathcal{H} \wedge v, \delta\mathcal{H}\big\rangle + \big\langle * \delta\mathcal{H} \wedge v, \iota_V * \mathcal{H}\big\rangle \\
&\quad - \Big[\big\langle \iota_V * \mathcal{H}, \iota_V(*\mathcal{H} - \mathcal{H} - 2\kappa(\bar{\lambda}, \gamma_{(3)}\lambda))\big\rangle - \big\langle \iota_V(*\mathcal{H} - \mathcal{H} - 2\kappa(\bar{\lambda}, \gamma_{(3)}\lambda)), \\
&\qquad \iota_V\mathcal{H}\big\rangle\Big] \wedge v \wedge \delta v + \delta_\lambda(\mathcal{L}'_{\text{PST}})|_{\mathcal{H}}
\end{aligned}$$

$$
\begin{aligned}
&\overset{(B.4)}{=} 2\big\langle \iota_V \mathscr{H}\wedge v, \delta\mathcal{H}\big\rangle + \big\langle \mathcal{H}, \delta\mathcal{H}\big\rangle - \big[\big\langle \iota_V(*\mathcal{H} - \mathcal{H} - \kappa(\bar{\lambda}, \gamma_{(3)}\lambda)) \\
&\qquad \wedge v\wedge\delta v, \iota_V(*\mathcal{H} - \mathcal{H} - \kappa(\bar{\lambda}, \gamma_{(3)}\lambda))\big\rangle + \delta_\lambda(\mathcal{L}'_{\rm PST})|_{\mathcal{H}} \\
&= 2\big\langle \iota_V\mathscr{H}\wedge v, \delta\mathcal{H} - \tfrac{1}{2}\delta v\wedge \iota_V\mathscr{H}\big\rangle + \big\langle \mathcal{H}, \delta\mathcal{H}\big\rangle + \delta_\lambda(\mathcal{L}'_{\rm PST})|_{\mathcal{H}}\ ,
\end{aligned}
$$

where we also use the fact that $\iota_V v = 1$ implies $\iota_V \delta v = 0$. Substituting the variation (B.11) of $\mathcal{H}$ then leads to

$$
\begin{aligned}
\delta(\mathcal{L}'_{\rm PST})|_{\mathcal{H}} &= 2\,\langle \iota_V\mathscr{H}\wedge v, {\rm d}(\Delta B) - 2\kappa(\mathcal{F}, \delta A) + \mu_1(\Delta C)\rangle - \langle \iota_V\mathscr{H}, \iota_V\mathscr{H}\rangle \wedge v\wedge\delta v \\
&\quad + \langle \mathcal{H}, {\rm d}(\Delta B) - 2\kappa(\mathcal{F}, \delta A) + \mu_1(\Delta C)\rangle + \delta_\lambda(\mathcal{L}'_{\rm PST})|_{\mathcal{H}} \\
&\underset{(5.32)}{\overset{(5.14)}{=}} \langle \mu_1(2\iota_V\mathscr{H}\wedge v + \mathcal{H}), \Delta C\rangle + \langle 2{\rm d}(\iota_V\mathscr{H}\wedge v) - \kappa(\mathcal{F},\mathcal{F}) + \mu_1(\mathscr{G}), \Delta B\rangle \\
&\quad - \big\langle \kappa(\mathcal{F}, 2\iota_V\mathscr{H}\wedge v + \mathcal{H}), \delta A\big\rangle - \langle \iota_V\mathscr{H}, \iota_V\mathscr{H}\rangle\wedge v\wedge\delta v + \delta_\lambda(\mathcal{L}'_{\rm PST})|_{\mathcal{H}}
\end{aligned}
$$

The remaining term in $\mathcal{L}_{\rm PST}$ leads to the variation

$$
\begin{aligned}
\delta(\mathcal{L}_{\rm PST})|_{\mathscr{G}} &\overset{(B.7)}{=} \big\langle \Phi(*(\delta\mathscr{G}\wedge v)), \mathscr{G}\wedge v\big\rangle + \big\langle \Phi(*(\mathscr{G}\wedge\delta v)), \mathscr{G}\wedge v\big\rangle + \delta_\phi(\mathcal{L}'_{\rm PST})_{\mathscr{G}} \\
&\quad + \big\langle \Phi(*(\mathscr{G}\wedge v)), \delta\mathscr{G}\wedge v\big\rangle + \big\langle \Phi(*(\mathscr{G}\wedge v)), \mathscr{G}\wedge\delta v\big\rangle + \delta_\phi(\mathcal{L}'_{\rm PST})_{\mathscr{G}} \\
&\overset{(B.8)}{=} 2\big\langle \Phi(*(\mathscr{G}\wedge v)), \delta\mathscr{G}\wedge v\big\rangle + 2\big\langle \Phi(*(\mathscr{G}\wedge v)), \mathscr{G}\wedge\delta v\big\rangle + \delta_\phi(\mathcal{L}'_{\rm PST})_{\mathscr{G}} \\
&\underset{(B.7)}{\overset{(B.6)}{=}} 2\big\langle \Phi(\iota_V *\mathscr{G}), \delta\mathscr{G}\wedge v\big\rangle - 2\big\langle \Phi(\iota_V *\mathscr{G}), \iota_V\mathscr{G}\big\rangle\wedge v\wedge\delta v + \delta_\phi(\mathcal{L}'_{\rm PST})_{\mathscr{G}} \\
&\overset{(5.32)}{=} 2\big\langle \Phi(\iota_V *\mathscr{G}), \delta\mathcal{G}\wedge v\big\rangle - 2\big\langle \kappa(\Phi(\iota_V *\mathscr{G}), \phi), *\delta\mathcal{F}\big\rangle\wedge v \\
&\quad - 2\big\langle \Phi(\iota_V *\mathscr{G}), \iota_V\mathscr{G}\big\rangle\wedge v\wedge\delta v + \delta_{\phi,\lambda,\chi}(\mathcal{L}'_{\rm PST})_{\mathscr{G}} \\
&\underset{(B.8)}{\overset{(B.4)}{=}} 2\big\langle \Phi(\iota_V *\mathscr{G}\wedge v), \delta\mathcal{G}\big\rangle - 2\big\langle \kappa(\Phi(\mathscr{G} + \iota_V\mathscr{G}\wedge v), \phi), \delta\mathcal{F}\big\rangle \\
&\quad - 2\big\langle \Phi(\iota_V *\mathscr{G}), \iota_V\mathscr{G}\big\rangle\wedge v\wedge\delta v + \delta_{\phi,\lambda,\chi}(\mathcal{L}'_{\rm PST})_{\mathscr{G}} \\
&\underset{(B.12)}{\overset{(B.10)}{=}} 2\big\langle \Phi(\iota_V *\mathscr{G}\wedge v), \nabla(\Delta C) + \kappa(\delta A, \mathcal{H}) + \kappa(\mathcal{F}, \Delta B) + \mu_1(\Delta D)\big\rangle
\end{aligned}
$$

$$
\begin{aligned}
&- 2\big\langle \kappa(\Phi(\mathscr{G} + \iota_V \mathscr{G} \wedge v), \phi), \nabla \delta A + \mu_1(\Delta B)\big\rangle \\
&- 2\big\langle \Phi(\iota_V * \mathscr{G}), \iota_V \mathscr{G}\big\rangle \wedge v \wedge \delta v + \delta_{\phi,\lambda,\chi}(\mathcal{L}'_{\mathrm{PST}})_{\mathscr{G}} \\
= \;& 2\big\langle \Phi(\iota_V * \mathscr{G} \wedge v), \mu_1(\Delta D)\big\rangle - 2\big\langle \nabla\Phi(\iota_V * \mathscr{G} \wedge v), \Delta C\big\rangle \\
&+ 2\big\langle \kappa(\Phi(\iota_V * \mathscr{G} \wedge v), \mathcal{F}), \Delta B\big\rangle - 2\big\langle \kappa(\Phi(\iota_V * \mathscr{G} \wedge v), \mathcal{H}), \delta A\big\rangle \\
&+ 2\big\langle \nabla(\kappa(\Phi(\mathscr{G} + \iota_V \mathscr{G} \wedge v), \phi)), \delta A\big\rangle - 2\big\langle \Phi(\iota_V * \mathscr{G}), \iota_V \mathscr{G}\big\rangle \wedge v \wedge \delta v \\
&+ \delta_{\phi,\lambda,\chi}(\mathcal{L}'_{\mathrm{PST}})_{\mathscr{G}} \; .
\end{aligned}
$$

Additionally, the variation of $\mathcal{L}_{\mathrm{top}}$ gives

$$
\begin{aligned}
\delta\mathcal{L}_{\mathrm{top}} &= \big\langle \mu_1(\delta C), \mathcal{H}\big\rangle + \big\langle \mu_1(C), \delta\mathcal{H}\big\rangle + \big\langle \delta B, \kappa(\mathcal{F},\mathcal{F})\big\rangle + 2\big\langle B, \kappa(\mathcal{F}, \delta\mathcal{F})\big\rangle \\
&\overset{(B.10)}{\underset{(B.11)}{=}} -\big\langle \mu_1(\mathcal{H}), \Delta C\big\rangle + \big\langle \mu_1(C), \mathrm{d}(\Delta B) - 2\kappa(\mathcal{F}, \delta A) + \mu_1(\Delta C)\big\rangle \\
&\quad + \big\langle \kappa(\mathcal{F},\mathcal{F}), \Delta B\big\rangle + \big\langle \kappa(\mathcal{F}, B), \nabla\delta A + \mu_1(\Delta B)\big\rangle \\
&= -\big\langle \mu_1(\mathcal{H}), \Delta C\big\rangle + \big\langle \kappa(\mathcal{F},\mathcal{F}) + \mu_1(\mathcal{G}), \Delta B\big\rangle - \big\langle \kappa(\mathcal{F},\mathcal{H}), \delta A\big\rangle \; .
\end{aligned}
$$

There are additional contributions to the variation with respect to δA in $\mathcal{L}_{\mathrm{tensor}}$ and $\mathcal{L}_{\mathrm{hyper}}$, which are straightforward to compute. Adding these to the sum of $\delta(\mathcal{L}'_{\mathrm{PST}})|_{\mathcal{H}}$, $\delta(\mathcal{L}_{\mathrm{PST}})|_{\mathscr{G}}$ and $\delta\mathcal{L}_{\mathrm{top}}$ then yields equation (5.45). The remaining terms in the full variation are also straightforward to compute as long as one takes care to use (B.3) to simplify the terms involving fermionic variations. The full variation is given by

$$
\begin{aligned}
\delta\mathcal{L} = \;& 2\big\langle \Phi(\iota_V * \mathscr{G} \wedge v), \mu_1(\Delta D)\big\rangle + \big\langle 2\mu_1(\iota_V \mathscr{H} \wedge v) - 2\nabla\Phi(\iota_V * \mathscr{G}) \wedge v, \Delta C\big\rangle \\
&+ \big\langle 2\mathrm{d}(\iota_V \mathscr{H} \wedge v) + 2\mu_1(\mathscr{G}) + 4\kappa(\Phi(\iota_V * \mathscr{G} \wedge v), \mathcal{F}), \Delta B\big\rangle \\
&+ \big\langle \mu_1(\mathscr{I}) - 2\kappa(\mathcal{F}, \iota_V \mathscr{H} \wedge v) + 2\nabla(\kappa(\Phi(\iota_V \mathscr{G} \wedge v), \phi)) \\
&- 2\kappa(\Phi(\iota_V * \mathscr{G} \wedge v), \mathcal{H}), \delta A\big\rangle - \big(\big\langle \iota_V \mathscr{H}, \iota_V \mathscr{H}\big\rangle + 2\big\langle \Phi(\iota_V * \mathscr{G}), \iota_V \mathscr{G}\big\rangle\big) \wedge v \wedge \delta v \\
&+ \big\langle 2\mathrm{d} * \mathrm{d}\phi - 2\kappa(\mathcal{F}, *\mathcal{F}) + 2\kappa(Y_{ij}, Y^{ij})\,\mathrm{vol} - 4\kappa(\bar\lambda, \nabla\!\!\!/\,\lambda)\,\mathrm{vol}
\end{aligned}
$$

$$
\begin{aligned}
&- 2\kappa(\Phi(\mathscr{G} + \iota_V \mathscr{G} \wedge v), \mathcal{F}) + \tfrac{1}{\phi_s}\langle \Phi(\iota_V * \mathscr{G}), *(\iota_V * \mathscr{G})\rangle, \delta\phi \big\rangle \qquad\qquad (B.13)\\
&+ \big\langle \delta\bar{\chi}^i, -8\not{d}\chi_i \,\mathrm{vol} + 4\kappa(\mathcal{F}, *\gamma_{(2)}\lambda_i) + 16\kappa(Y_{ji}, \lambda^j)\,\mathrm{vol} - 4\kappa(\Phi(\iota_V * \mathscr{G} \wedge v), *\gamma_{(2)}\lambda_i)\big\rangle\\
&+ \big\langle \delta\bar{\lambda}^i, -4\kappa(\not{\nabla}\lambda_i, \phi)\,\mathrm{vol} - 2\kappa(\lambda_i, \not{d}\phi)\,\mathrm{vol} + 2\kappa(\mathcal{H}, *\gamma_{(2)}\chi_i) - 4\kappa(Y_{ij}, \chi^j)\,\mathrm{vol}\\
&- \tfrac{1}{4}\kappa(*\gamma_{(3)}\lambda_i, \mathcal{H}) + \mu_1(\prec - \rhd q_i, \psi \succ)\,\mathrm{vol} + 2\kappa(\Phi(\iota_V * \mathscr{G}), *\gamma_{(2)}\chi_i)\big\rangle\\
&+ \big\langle 4\kappa(Y^{ij}, \phi)\,\mathrm{vol} - 8\kappa(\bar{\lambda}^i, \chi^j)\,\mathrm{vol} + \tfrac{1}{2}\mu_1(\prec q^{(i}, - \rhd q^{j)} \succ)\,\mathrm{vol}, \delta Y_{ij}\big\rangle\\
&+ \prec \nabla * \nabla q^i + 4\bar{\lambda}^i \rhd \psi\,\mathrm{vol} + 2Y^i{}_j \rhd q^j\,\mathrm{vol}, \delta q_i \succ\\
&+ \prec \delta\bar{\psi}, 2\not{\nabla}\psi\,\mathrm{vol} - 2\lambda \rhd q\,\mathrm{vol} \succ \quad ,
\end{aligned}
$$

which leads to the equations of motion in Section 5.6. Using this expression one can immediately verify that the Lagrangian is invariant under the symmetry transformations given in (5.46). We can now also verify that the transformation (5.6) is indeed a symmetry:

$$
\begin{aligned}
\delta_{\varphi_v}\mathcal{L} &= \big\langle \Phi(\iota_V * \mathscr{G} \wedge v), \mu_1(\varphi_v \iota_V \mathscr{I})\big\rangle - \big\langle 2\mu_1(\iota_V \mathscr{H} \wedge v) - 2\nabla\Phi(\iota_V * \mathscr{G}) \wedge v, \varphi_v \iota_V \mathscr{G}\big\rangle\\
&\quad + \big\langle 2\mathrm{d}(\iota_V \mathscr{H} \wedge v) + 2\mu_1(\mathscr{G}) + 4\kappa(\Phi(\iota_V * \mathscr{G} \wedge v), \mathcal{F}), \varphi_v \iota_V \mathscr{H}\big\rangle\\
&\quad + \big\langle \mu_1(\mathscr{I}) - 2\kappa(\mathcal{F}, \iota_V \mathscr{H} \wedge v) + 2\nabla(\kappa(\Phi(\iota_V \mathscr{G} \wedge v), \phi)), \varphi_v \Phi(\iota_V * \mathscr{G})\big\rangle\\
&\quad - \big(\big\langle \iota_V \mathscr{H}, \iota_V \mathscr{H}\big\rangle + 2\big\langle \Phi(\iota_V * \mathscr{G}), \iota_V \mathscr{G}\big\rangle\big) \wedge v \wedge \mathrm{d}\varphi_v\\
&= \big\langle \Phi(\iota_V * \mathscr{G} \wedge v), \mu_1(\varphi_v \iota_V \mathscr{I})\big\rangle - \big\langle 2\mu_1(\iota_V \mathscr{H} \wedge v) - 2\nabla\Phi(\iota_V * \mathscr{G}) \wedge v, \varphi_v \iota_V \mathscr{G}\big\rangle\\
&\quad + \big\langle 2\mathrm{d}(\iota_V \mathscr{H}) \wedge v + 2\mu_1(\mathscr{G}) + 4\kappa(\Phi(\iota_V * \mathscr{G} \wedge v), \mathcal{F}), \varphi_v \iota_V \mathscr{H}\big\rangle\\
&\quad + \big\langle \mu_1(\mathscr{I}) - 2\kappa(\mathcal{F}, \iota_V \mathscr{H} \wedge v) + 2\nabla(\kappa(\Phi(\iota_V \mathscr{G} \wedge v), \phi)), \varphi_v \Phi(\iota_V * \mathscr{G})\big\rangle\\
&\quad - \big\langle 2\mathrm{d}(\iota_V \mathscr{H}) \wedge v, \varphi_v \iota_V \mathscr{H}\big\rangle - 2\big\langle \nabla\Phi(\iota_V * \mathscr{G}) \wedge v, \varphi_v \iota_V \mathscr{G}\big\rangle\\
&\quad - 2\big\langle \Phi(\iota_V * \mathscr{G}) \wedge v, \varphi_v \nabla \iota_V \mathscr{G}\big\rangle\\
&\overset{(B.6)}{=} \big\langle \Phi(\iota_V * \mathscr{G} \wedge v), \mu_1(\varphi_v \iota_V \mathscr{I})\big\rangle + \big\langle \mu_1(\mathscr{I}), \varphi_v \Phi(\iota_V * \mathscr{G})\big\rangle
\end{aligned}
$$

$$
\begin{aligned}
&\overset{\text{(B.6)}}{=} \Big\langle \Phi(\iota_V * \mathscr{G} \wedge v), \mu_1(\varphi_v \iota_V \mathscr{I})\Big\rangle - \Big\langle \mu_1(\iota_V \mathscr{I}), \varphi_v \Phi(\iota_V * \mathscr{G})\Big\rangle \wedge v \\
&= \; 0 \;.
\end{aligned}
$$

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